Fractal Geometry and its applications

V. Drakopoulos

Department of Computer Science and Biomedical Informatics

UNIVERSITY OF THESSALY
FOUNDED 1984

Bibliography

- Barnsley M. F., *Fractals everywhere*, 3rd ed., Dover Publications, Inc., New York, 2012.
- Barnsley M. F., *SuperFractals*, Cambridge University Press, New York, 2006.
- Barnsley M. F. and Anson, L. F., *The fractal transform*, Jones and Bartlett Publishers, Inc, 1993.
- Barnsley M. F. and Hurd L. P., *Fractal image compression*, AK Peters, Wellesley, 1992.
- Barnsley M. F., Saupe D. and Vrscay E. R. (eds.), *Fractals in multimedia*, Springer-Verlag, New York, 2002.
- Beardon A. F., *Iteration of rational functions*, Springer-Verlag, New York, 1991.
- Carleson L. and Gamelin T., *Complex dynamics*, Springer-Verlag, New York, 1993.
- Crownover R. M., *Introduction to fractals and chaos*, Jones and Bartlett Publishers, Boston, 1995.
- Devaney R. L., *An introduction to chaotic dynamical systems*, 3rd ed., Addison-Wesley, Reading, MA, 2021.
- Encarnação J. L., Peitgen H.-O, Sakas G. and Englert Gabriele (eds.), *Fractal geometry and computer graphics*, Springer-Verlag, 1992.
- Falconer K. J., *Fractal geometry: Mathematical foundations and applications*, 3rd ed., Wiley, Chichester, 2014.
- Fisher Y., *Fractal image compression* (ed.), Springer-Verlag, New York, 1995.
- Hoggar S. G., *Mathematics for computer graphics*, Cambridge University Press, Cambridge, 1992.
- Lu N., *Fractal imaging*, Academic Press, San Diego, CA, 1997.

Bibliography

- Mandelbrot B. B., *Fractals: Form, chance and dimension*, W. H. Freeman, San Francisco, 1977.
- Mandelbrot B. B., *The fractal geometry of nature*, W. H. Freeman, New York, 1982.
- Massopust P. R., *Fractal functions, fractal surfaces and wavelets*, 2nd ed., Academic Press, San Diego, CA, 2016.
- Massopust P. R., *Interpolation and approximation with splines and fractals*, Oxford University Press, 2010.
- Mc Mullen C., *Complex dynamics and renormalization*, Princeton Univ. Press, Princeton, NJ, 1994.
- Nikiel S., *Iterated function systems for real-time image synthesis*, Springer-Verlag, London, 2007.
- Peitgen H.-O., Jürgens H. and Saupe D., *Fractals for the classroom*, Springer-Verlag, 1992.
- Peitgen H.-O. and Richter P. H., *The beauty of fractals*, Springer-Verlag, New York, 1986.
- Peitgen H.-O. and Saupe D. (eds.), *The science of fractal images*, Springer-Verlag, New York, 1988.
- Steinmetz N., *Rational iteration*, de Gruyter, Berlin, 1993.

Bibliography

- Μπούντης Αν. Χ., *Δυναμικά συστήματα & χάος*, Τόμος Α’. Βούλγαρη, 1989.
- Μπούντης Αν., *Δυναμικά συστήματα και χάος*, Τόμος Α. Παπασωτηρίου, 1995.
- Ευαγγελάτου-Δάλλα Λεώνη, *Στοιχεία fractal γεωμετρίας*, Τμήμα Μαθηματικών Ε.Κ.Π.Α., 2000.
- Αραχωβίτης Ιωάν., *Εισαγωγή στη χαοτική δυναμική και στα fractals (κλασμοειδή)*, Εκδόσεις Παπασωτηρίου, 2001.
- Μπούντης Τ., *Ο θαυμαστός κόσμος των fractal*, Leader Books, 2004.
- Bak P., *Πως λειτουργεί η φύση: Η επιστήμη της αυτοοργανούμενης κρισιμότητας*, Εκδόσεις Κάτοπτρο, 2008.
- Αναστασίου Στ. και Μπούντης Αν., *Συνεχή και διάκριτα δυναμικά συστήματα και μία εισαγωγή στη θεωρία του χάους*, Β’ Έκδ. Α. Γ. Πνευματικός, 2020.

Outline

Outline

1. PROLOGUE

- Fractality
- Determinism
- Chaoticity

Introduction

The investigation of the meaning of words is the beginning of education.

Antisthenes

- *Fractal* comes from the Latin adjective *fractus*, which has the same root as *fraction* and *fragment*, and means "irregular or fragmented;" it is related to *frangere*, which means "to break."

Determinism and Chaos

- Like the queen of England, determinism reigns but does not govern.

Michael Berry (1988),
Professor in the Department of Physics,
University of Bristol

- Χάος: Ἡ ἄβυσσος, μέγα βάραθρον || το άπειρο σκότος (διάστημα)· (μτφ) σύγχυση, αναστάτωση, ακαταστασία, αταξία.
- Χάος: Απρόβλεπτη κατάσταση στην οποία περιέρχεται ένα αιτιοκρατικό σύστημα λόγω ευαισθησίας στις αρχικές συνθήκες.
- Κατάσταση αδυναμίας πρόβλεψης, φαινομενικής τυχαιότητας της συμπεριφοράς ενός αιτιοκρατικού συστήματος.

Outline

Outline

2. INTRODUCTION

- Early history
- Classic fractals
- Space-filling curves

Introduction

Many natural and artificial phenomena

- have the very fundamental characteristic of invariance under different scales,
- have infinite details at every point,
- are self-similar across different scales and
- can be described by a procedure that specifies a repeated operation for producing the details.

The beginning

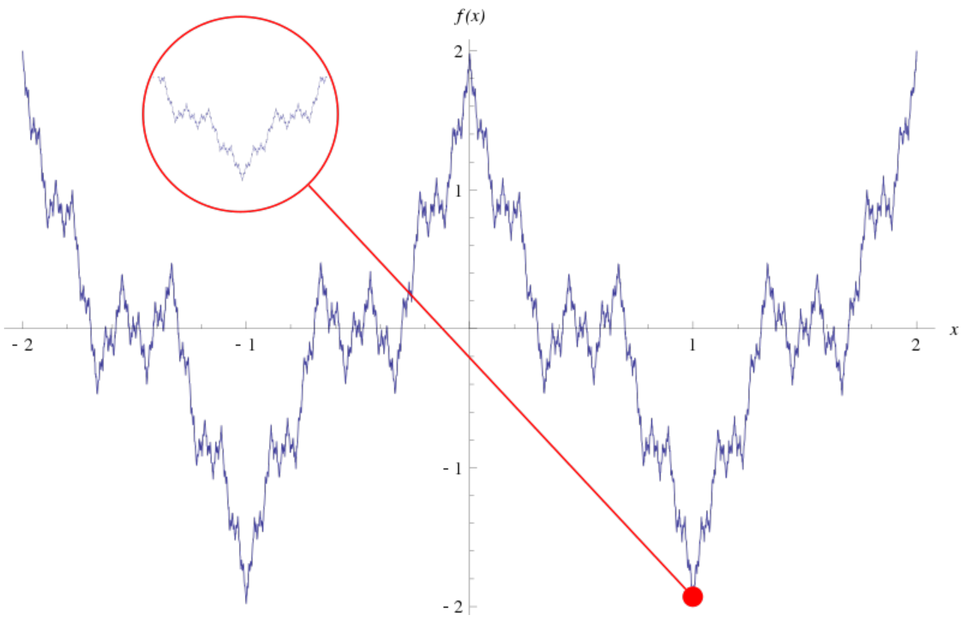
- Draw a line on a sheet of paper.
- Euclidean geometry tells us that this is a figure of one dimension, namely the length.
- Now extend the line.
- Make it wind around and around, back and forth, without crossing itself, until it fills the entire sheet of paper.
- Euclidean geometry says that this is still a line, a figure of one dimension.
- But our intuition strongly tells us that if the line completely fills the entire plane, it must be two-dimensional.

Genesis

- Such thinking started a revolution in mathematics about a hundred years ago.
- Mathematicians such as Georg Cantor, Giuseppe Peano, David Hilbert, Felix Hausdorff, Helge von Koch and Wacław Sierpiński drew curves that were called “monsters”, “psychotic” and “pathological” by traditional mathematicians.
- A new type of dimensioning was proposed, in which a curve could have a fractional dimension, not just an integer one.

Weierstrass function (1872)

- The Weierstrass function
  $$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$
  where $0 < a < 1$, b positive odd integer and $ab > 1 + 3/2\pi$ is an example of a pathological real-valued function on the real line.
- The function has the property of being continuous everywhere but differentiable nowhere.
- It is named after its discoverer Karl Weierstrass.



Plot of Weierstrass function over the interval [−2, 2]. Like some fractals, the function exhibits self-similarity: every zoom (red circle) is similar to the global plot.

Cantor (ternary) set (1883)

Discovered in 1874 by Henry John Stephen Smith and introduced by German mathematician Georg Cantor

- Initially, we consider the closed set $c_0 = [0, 1]$.
- Remove from c_0 its middle third. What remains is the set $c_1 = [0, 1/3] \cup [2/3, 1]$.
- Remove the middle third of [0, 1/3] and [2/3, 1].
- Continuing this ad infinitum, we get the Cantor set

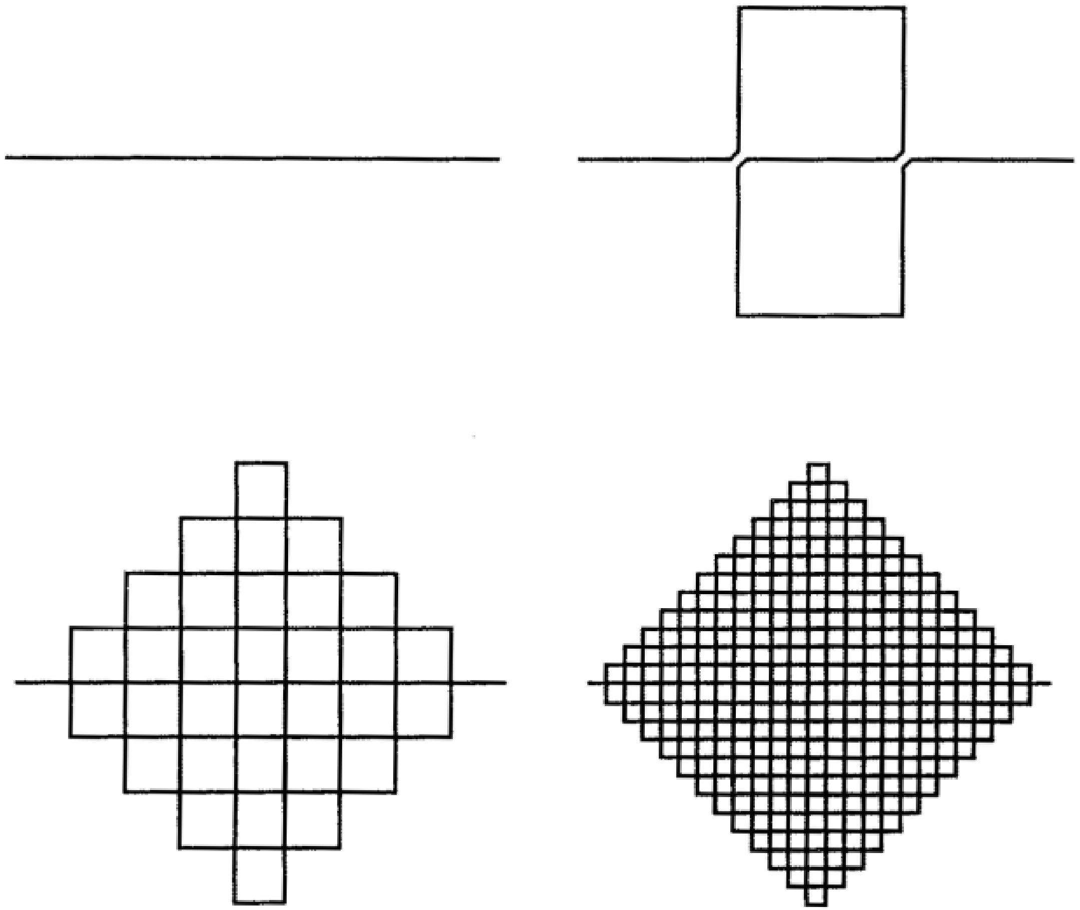
$$C = \bigcap_{n=0}^{\infty} c_n .$$

A brief history

- In 1890, Giuseppe Peano discovered a densely self-intersecting curve that passes through every point of the unit square.
- His purpose was to construct a continuous mapping from the unit interval onto the unit square.
- He was motivated by Georg Cantor's earlier counterintuitive result that the infinite number of points in a unit interval is the same cardinality as the infinite number of points in any finite-dimensional manifold, such as the unit square.
- The problem Peano solved was whether such a mapping could be continuous; i.e., a curve that fills a space.

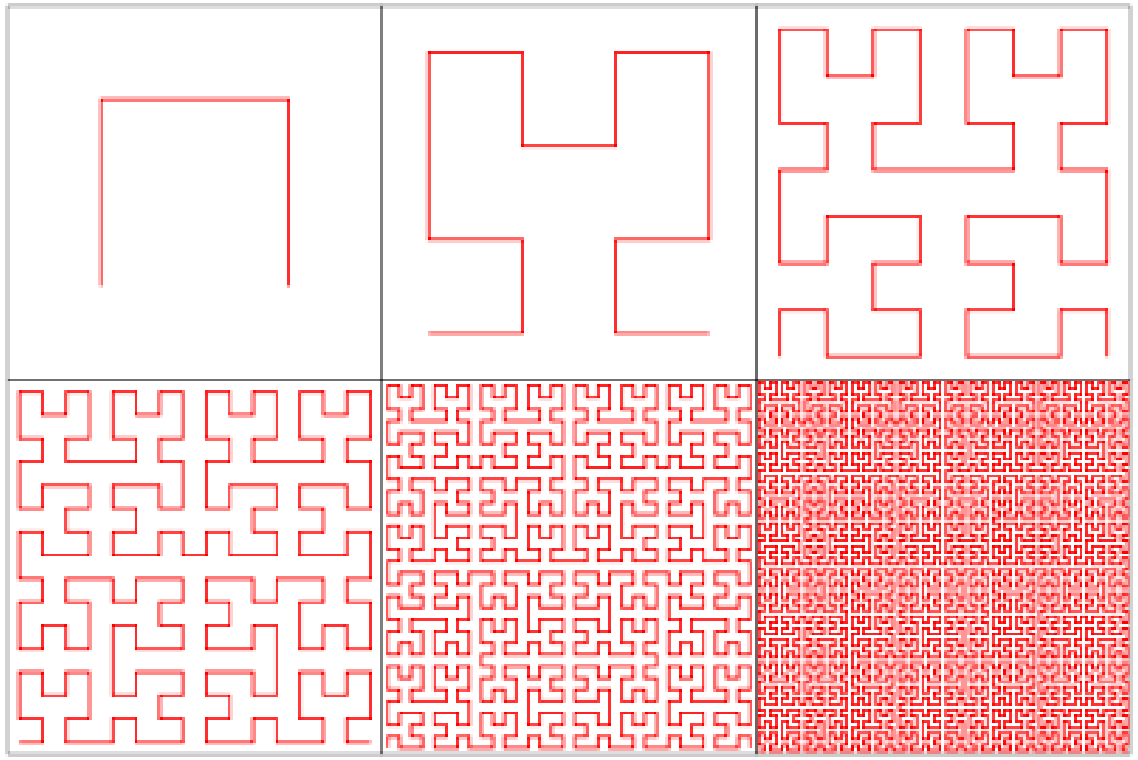
The Peano curve (1890)

Peano, G. “Sur une courbe, qui remplit toute une aire plane.” Math. Ann. 36 (1890), 157–160.



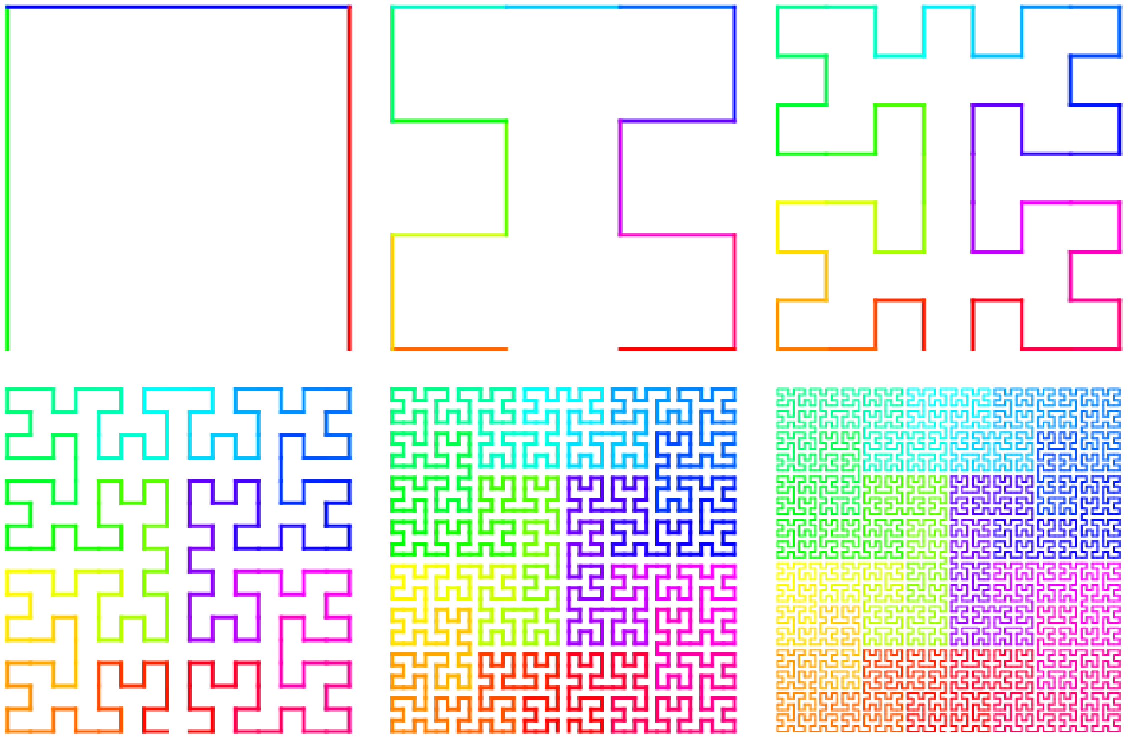
Hilbert curve in 2D (1891)

- A continuous fractal space-filling curve first described by the German mathematician David Hilbert in 1891 as a variant of the space-filling curves discovered by Giuseppe Peano in 1890.
- The first four iterations are shown on the right.
- D. Hilbert, “Über die stetige Abbildung einer Linie auf ein Flächenstück”. Math. Ann. 38 (1891), 459–460.



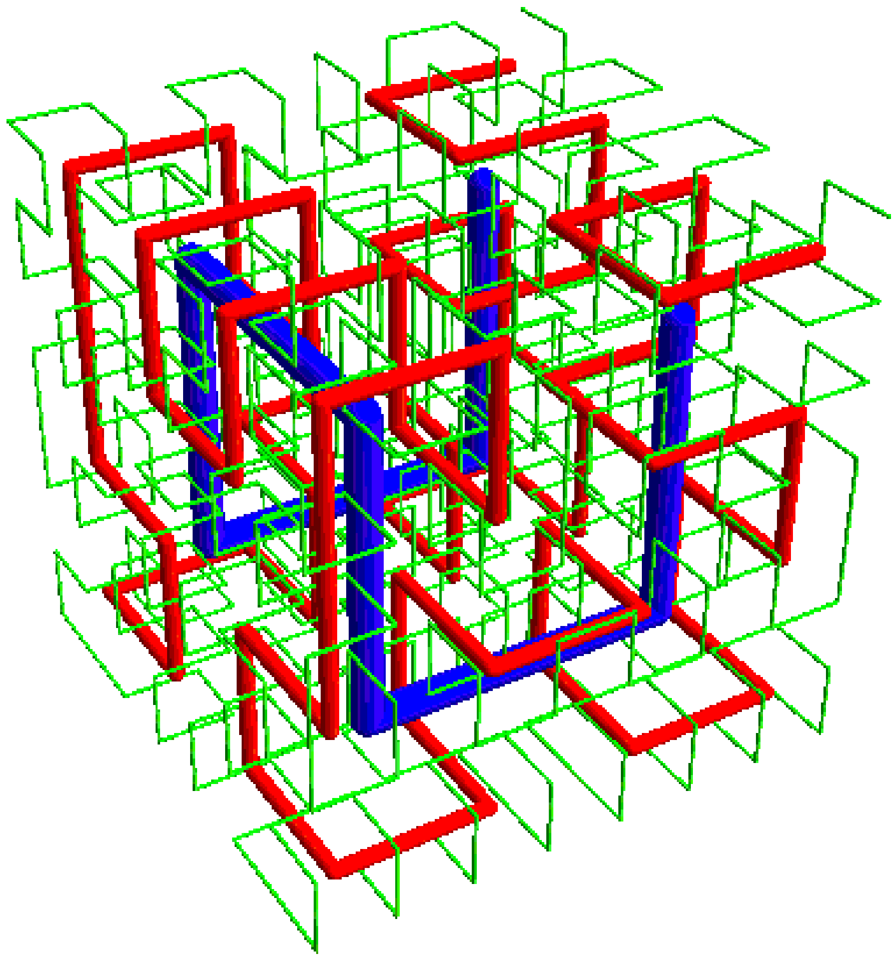
Moore curve (1900)

- One difference being that the start and end points are adjacent corners of the square in the Hilbert curve, and adjacent points in the Moore curve.
- The first six stages of the Moore curve are shown on the right.
- Moore, Eliakim Hastings. “On Certain Crinkly Curves.” Transactions of the AMS, vol. 1, no. 1 (1900), 72–90.



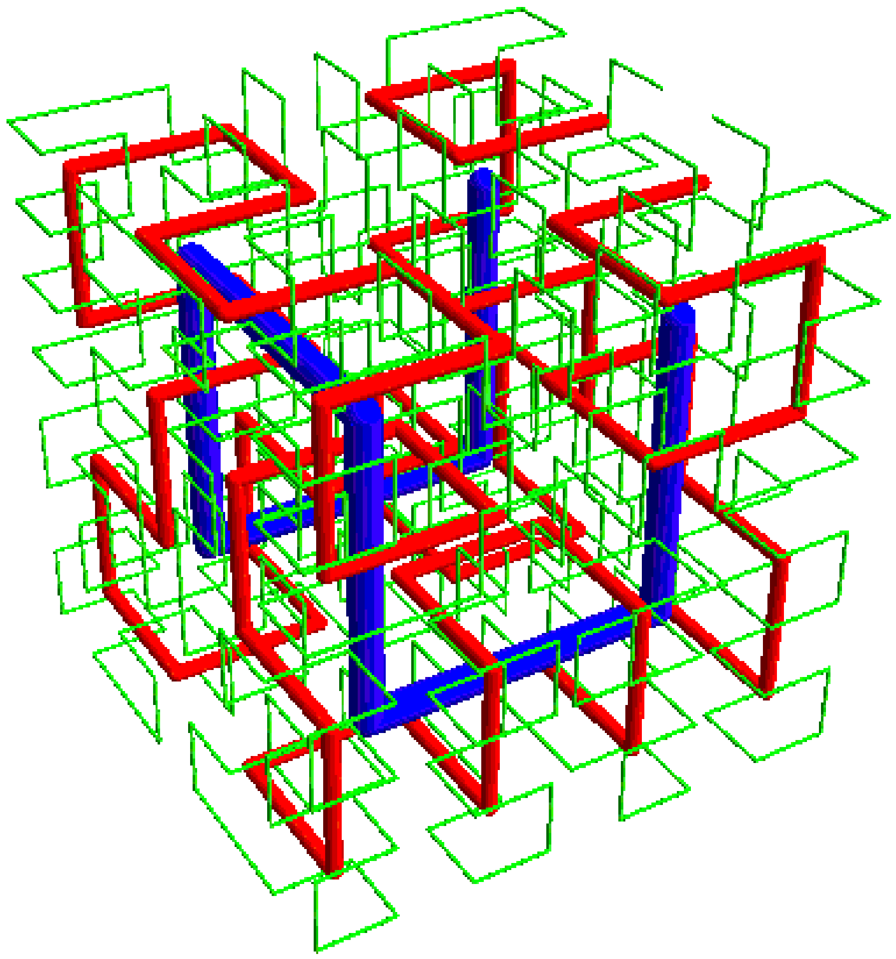
Hilbert curve in 3D

- The Hilbert curve as well as the Moore curve are two famous plane-filling curves that can be extended to 3D space-filling curves.
- A three-dimensional analog of the Hilbert curve is shown on the right; here with the first three iterations intertwined.

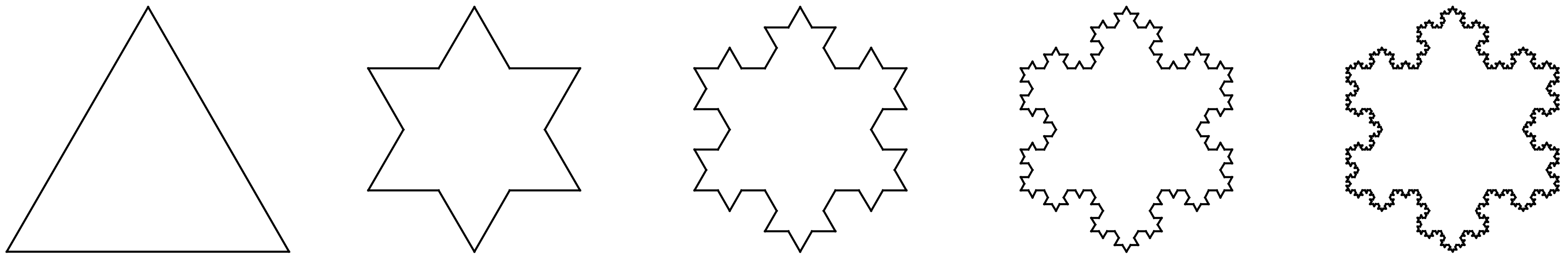


Moore curve in 3D

- The Hilbert curve as well as the Moore curve are two famous plane-filling curves that can be extended to 3D space-filling curves.
- A three-dimensional analog of the Moore curve is shown on the right; here with the first three iterations intertwined.
- Again, in the Hilbert curve the start and end are adjacent corners of the cube, while in the Moore case the ends are adjacent points.



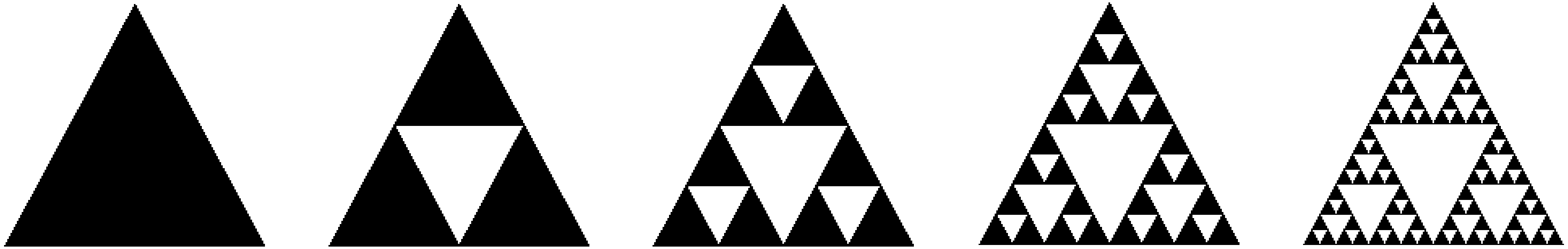
Koch snowflake (1904)



The first five iterations of the Koch snowflake

- The Koch curve can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:
  1. Divide the line segment into three segments of equal length.
  2. Draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
  3. Remove the line segment that is the base of the triangle from step 2.
- After one iteration of this process, the result is a shape similar to the Star of David.
- The Koch curve is the limit approached as the above steps are followed over and over again.

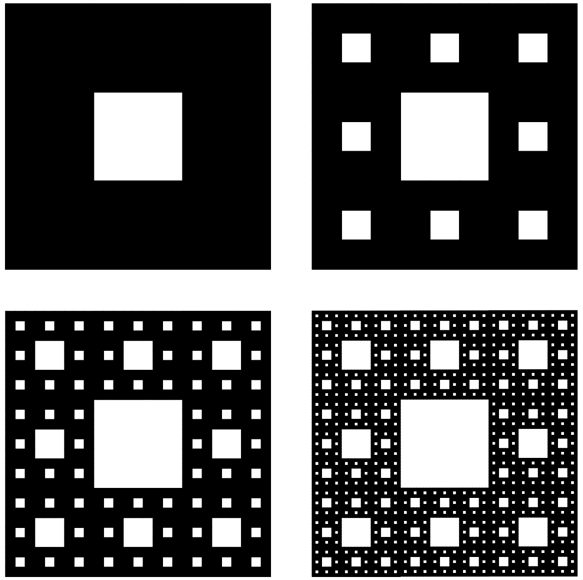
Sierpiński gasket (1915)



- Start with a solid (filled) equilateral triangle $S(0)$.
- Divide this into four smaller equilateral triangles using the midpoints of the three sides of the original triangle as the new vertices and remove the interior of the middle triangle to get $S(1)$.
- Repeat this procedure on each of the three remaining solid equilateral triangles to obtain $S(2)$ and continuing we get

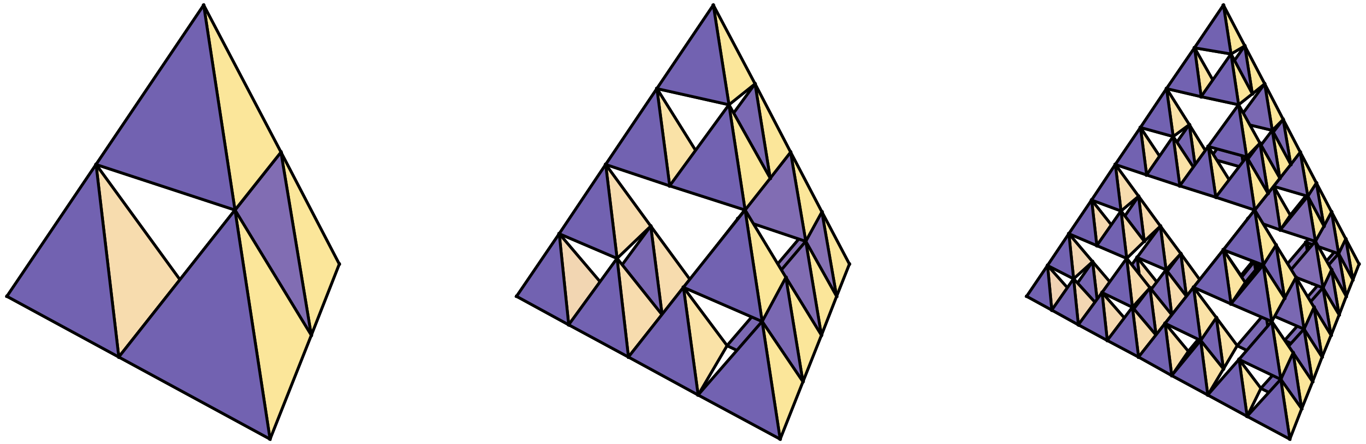
$$S = \bigcap_{i=1}^{\infty} S(i).$$

Sierpiński carpet (1916)



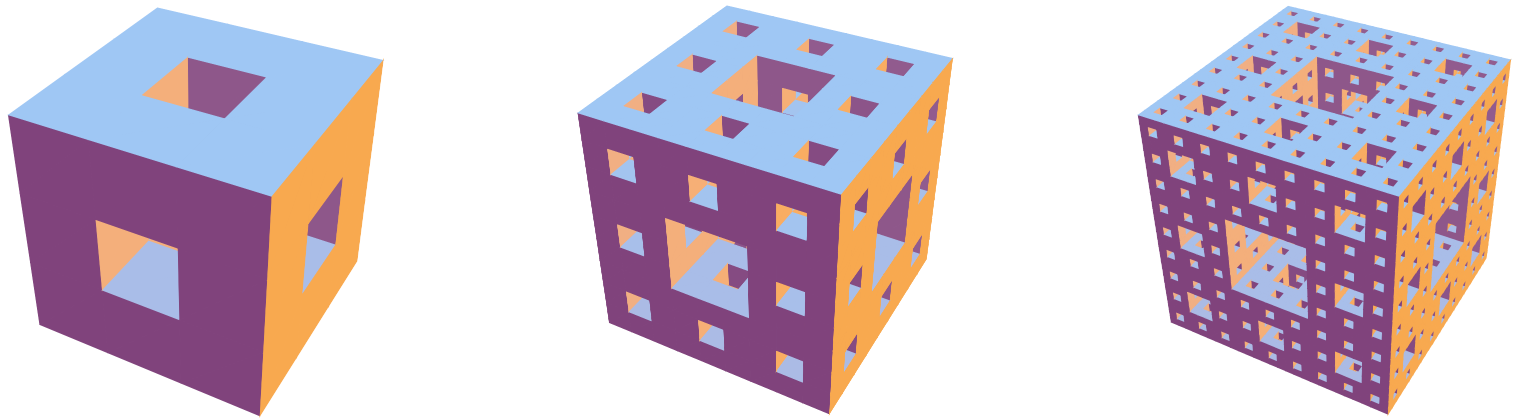
- Begin with a square.
- The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the central subsquare is removed.
- The same procedure is then applied recursively to the remaining 8 subsquares, *ad infinitum*.

Sierpiński tetrahedron



- The tetrix is the three-dimensional analogue of the Sierpiński triangle, formed by repeatedly shrinking a regular tetrahedron to one half its original height, putting together four copies of this tetrahedron with corners touching, and then repeating the process.
- This can also be done with a square pyramid and five copies instead.

Menger sponge (1926)



- Begin with a cube. (*first image*)
- Divide every face of the cube into 9 squares, like a Rubik's Cube. This will subdivide the cube into 27 smaller cubes.
- Remove the cube at the middle of every face and remove the cube in the center, leaving 20 cubes, resembling a Void Cube. (*second image*). This is a level-1 Menger sponge.
- Repeat steps 1–3 for each of the remaining smaller cubes.

Deterministic nonperiodic flow (1963)

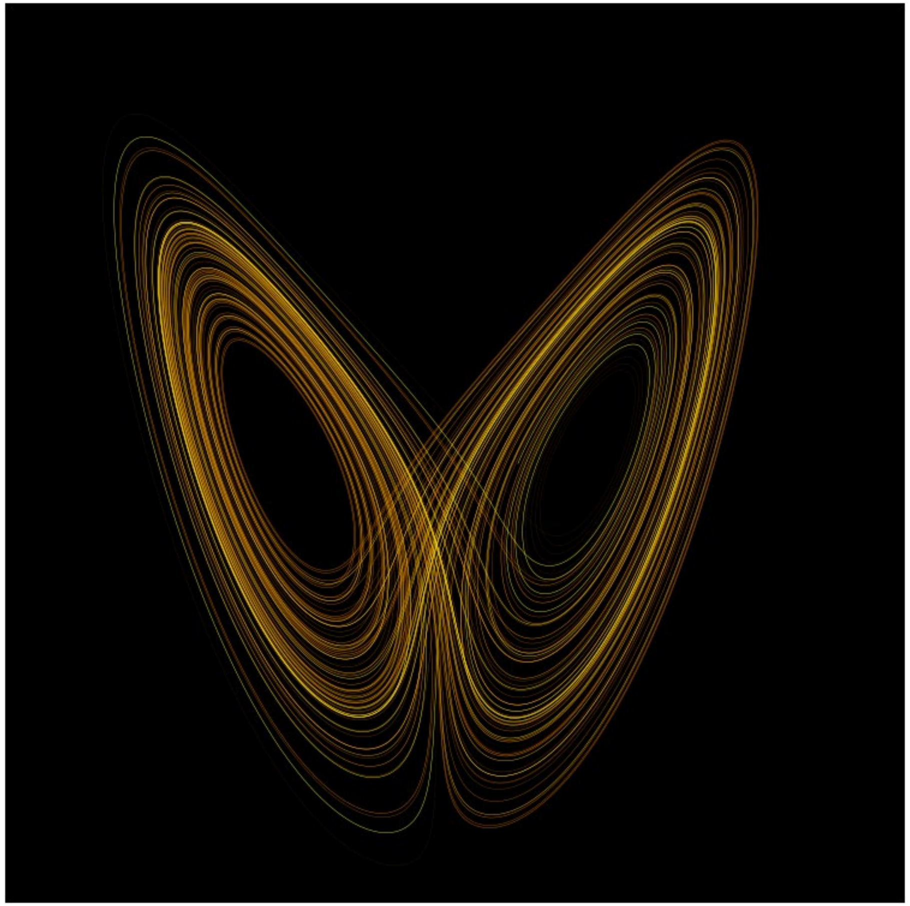
- In 1962, Edward Lorenz was attempting to develop a model of the weather when he observed some strange discrepancies in the behaviour of his model.
- In 1963 he described in his report a family of three ordinary differential equations with parameters a, b, c:

$$dx/dt = a(y - x)$$
$$dy/dt = bx - y - xz$$
$$dz/dt = xy - cz$$

(1972) Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?

Edward Norton Lorenz (1917–2008)

The butterfly effect



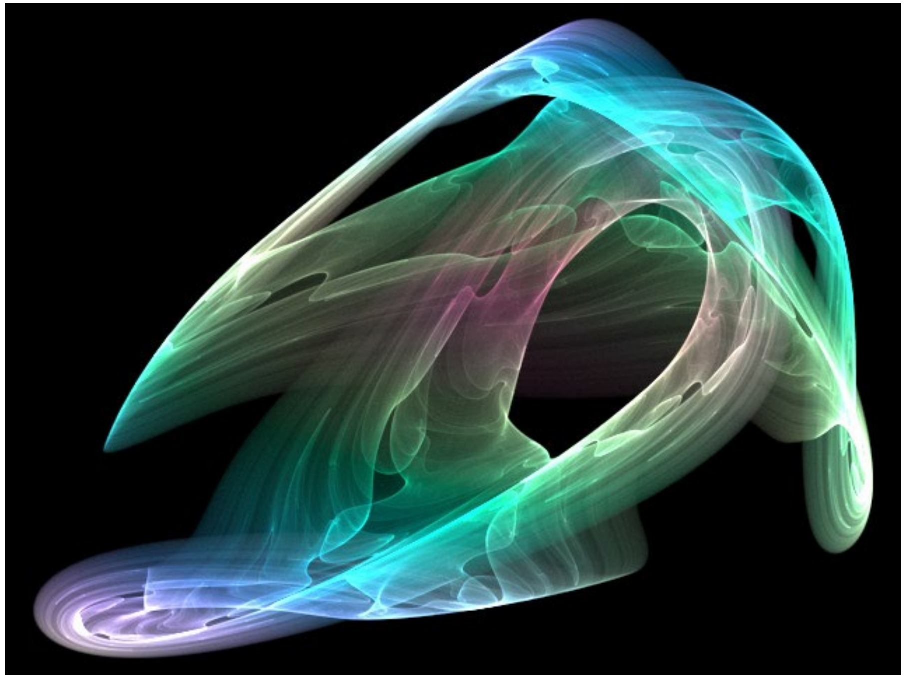
Lorenz attractor
(a = 10, b = 28, c = 8/3)

- Predictability: Different conclusions can be drawn from similar initial assumptions or conditions.
- The complicated correlations and dependencies of the parameters can finally be explained primarily through graphical methods.
- The Lorenz system of equations is perhaps the most well-known example of a continuous dynamical system with a fractal attractor.

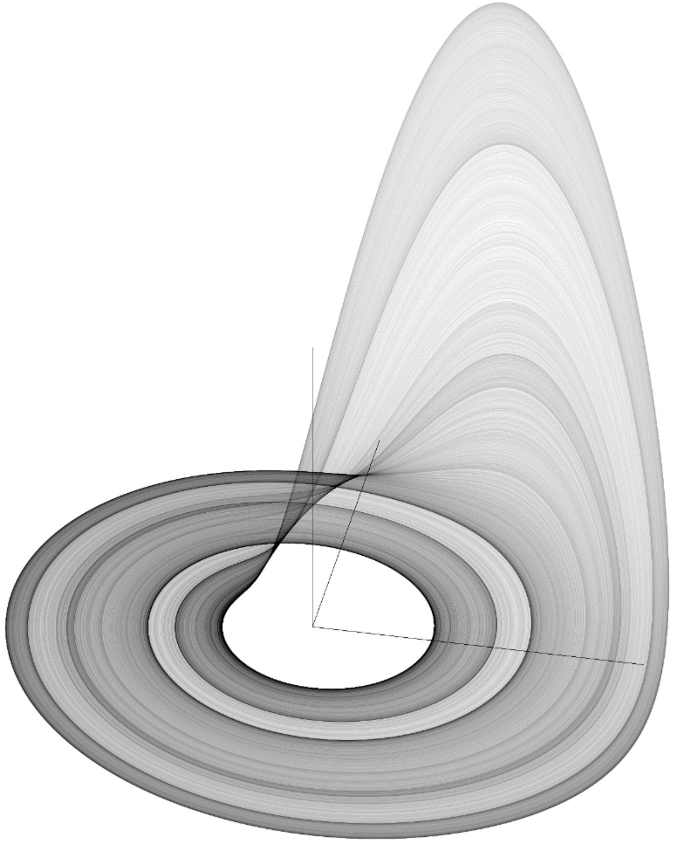
Sharkovsky's Theorem (1964)

- Order the natural numbers as follows:
$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec 13 \prec 15 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7$
$\prec 2 \cdot 9 \prec \cdots \prec 2 \cdot 2 \cdot 3 \prec 2 \cdot 2 \cdot 5 \prec 2 \cdot 2 \cdot 7$
$\prec 2 \cdot 2 \cdot 9 \prec \cdots \prec 2 \cdot 2 \cdot 2 \cdot 3 \prec \cdots \prec 2^5 \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1.$
- Now let F be a continuous function from the reals to the reals and suppose $p \prec q$ in the above ordering. Then, if F has a point of least period p, then F also has a point of least period q.
- A special case of this general result, also known as Sharkovsky's theorem, states that, if a continuous real function has a periodic point with period 3, then there is a periodic point of period n for every integer n.

Strange attractors (1971, 1976)



Ruelle, D., Takens, F. "On the nature of turbulence", Commun. Math. Phys. 20, 167–192 (1971)



Rössler, O. E. (1976), "An Equation for Continuous Chaos", Physics Letters, 57A (5): 397–398

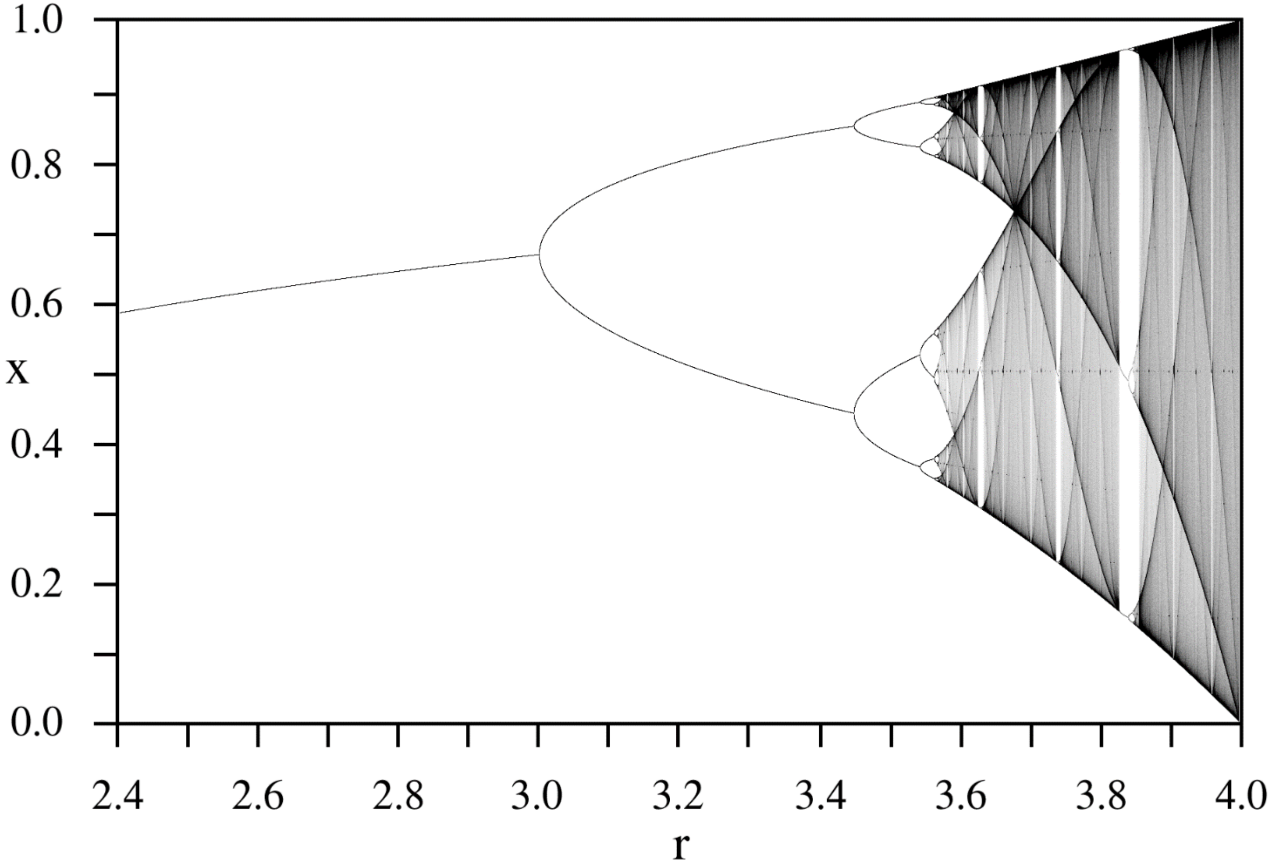
Characterization of Strange Attractors

- Let $T(x, y)$ be a given transformation in the plane with coordinates x and y. A bounded subset A of the plane is a chaotic and strange attractor for the transformation T, if there exists a set R with the following properties.
  - **Attractor.** R is a neighborhood of A, i.e., for each point in A there is a small disk centered at (x, y) which is contained in R. This implies in particular that A is in R. R is a trapping region, i.e., each orbit started in R remains in R for all iterations. Moreover, the orbit becomes close to A and stays as close to it as we desire. Thus, A is an *attractor*.
  - **Sensitivity.** Orbits started in R exhibit sensitive dependence on initial conditions. This makes A a *chaotic* attractor.
  - **Fractal.** The attractor has a fractal structure and is therefore called a *strange attractor*.
  - **Mixing.** A cannot be split into two different attractors. There are initial points in R with orbits that get arbitrarily close to any point of the attractor A.
- Strange *nonchaotic* attractors also exist; it is a form of attractor which, while converging to a limit, is strange, because it is not piecewise differentiable, and also non-chaotic, in that its Lyapunov exponents are non-positive.

Period three implies chaos (1975)

- Robert May's friend James A.Yorke did a rigorous mathematical analysis of the behaviour of the population equation and in December 1975, together with Tien–Yien Li published a paper.
- What Yorke and Li were able to show is that, if a function similar to the population equation has a period of three, then it has periods of every other number, n.
- There are two parts to the paper of Li and Yorke. First, that period three implies all other periods. This is a very special case of Sharkovsky's theorem. But Li and Yorke also proved that period three implies an uncountable number of non-periodic points, which is not part of Sharkovsky's paper.

$$x_n = r\,x_{n-1}\,(1 - x_{n-1})$$



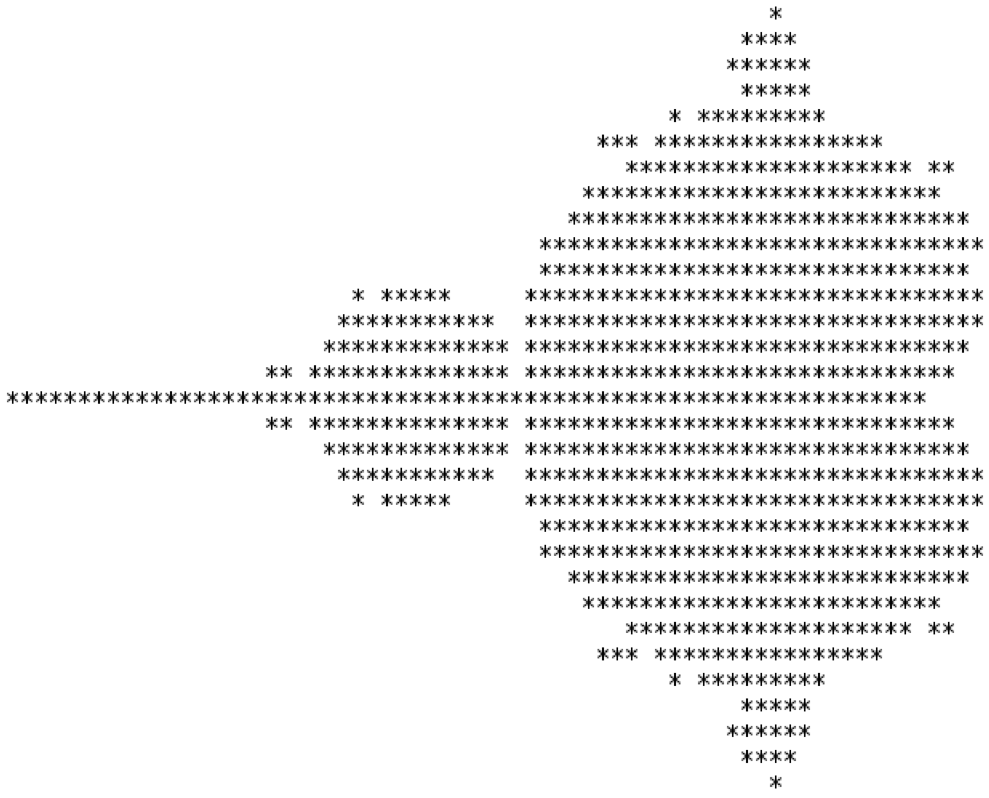
Final state (bifurcation) diagram for the population equation.

Feigenbaum constant (1975)

- Feigenbaum discovered in 1975, using an HP-65 calculator, that the ratio of the difference between the values at which such successive period-doubling bifurcations occur tends to a constant of around 4.6692...
- If Mitchell Feigenbaum had known of the work of Robert May and James Yorke, or if he had been able to view May's bifurcation diagrams, he might never have made his significant discovery.
- But, in 1976, Feigenbaum was looking at the population equation from a different point of view.

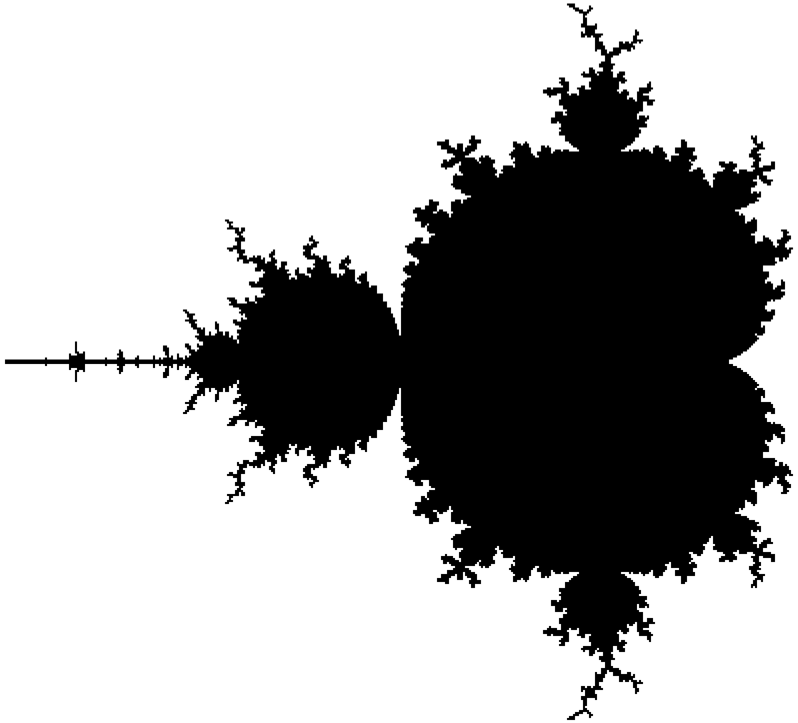
Robert W. Brooks and Peter Matelski (1978)

The first published picture of the Mandelbrot set as part of a study of Kleinian groups. The set of c's such that $f(z) = z^2 + c$ has a stable periodic orbit.



Mandelbrot set (1980)

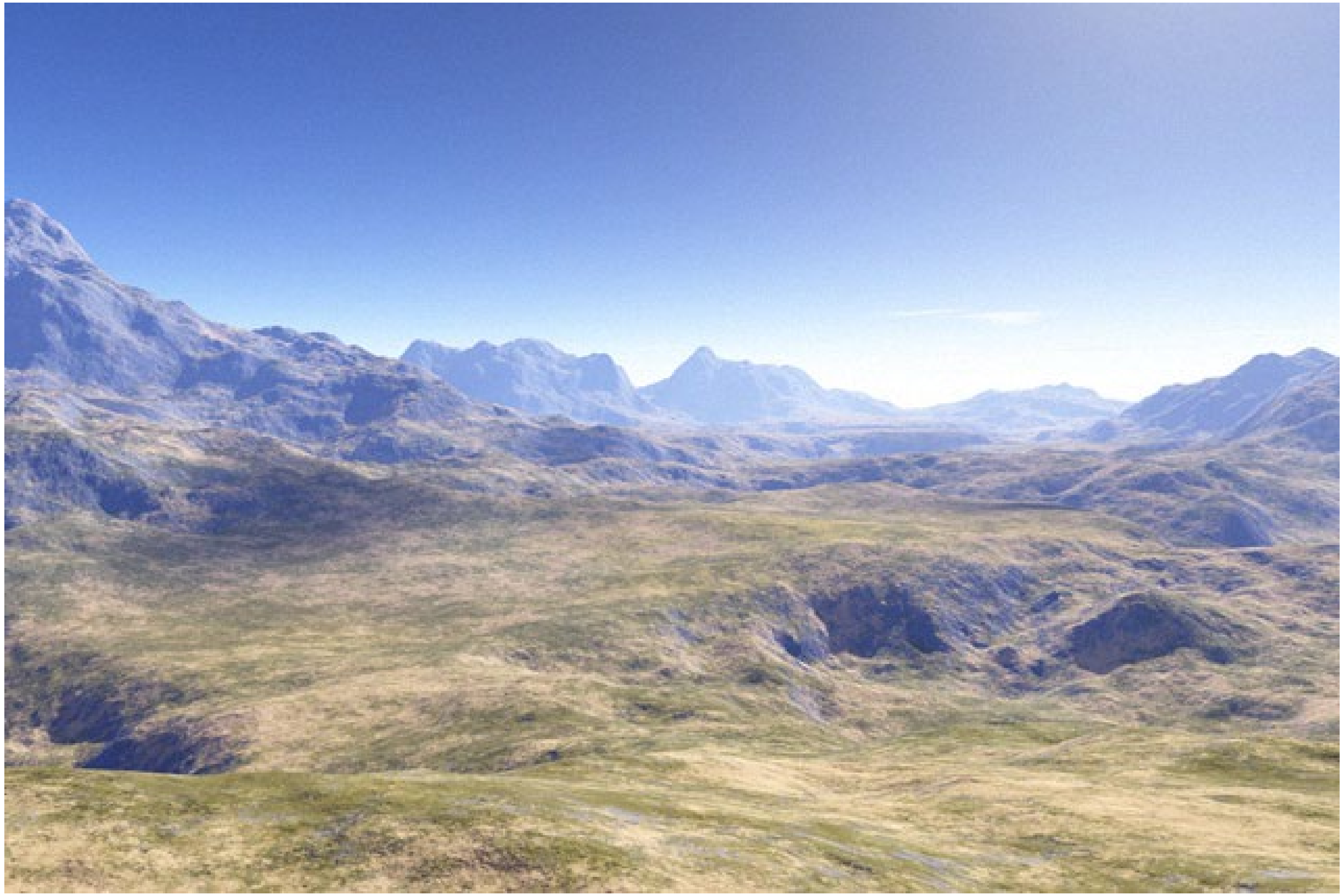
The set of complex values c that do not diverge under the squaring transform $p(z) = z^2 + c$ beginning with $z = 0$.



Barnsley fern (1988)

- It is a fractal named after the British mathematician Michael Barnsley who first described it in his book *Fractals Everywhere*.
- He made it to resemble the Black Spleenwort, *Asplenium adiantum-nigrum*.

Landscapes



Outline

Outline

1. Prologue
2. Introduction
3. On the dimension
4. Iterated function systems
5. Fractal interpolation
6. Complex analytic dynamics

3. ON THE DIMENSION

- General concept
- Metric spaces
- Self-similarity

The concept

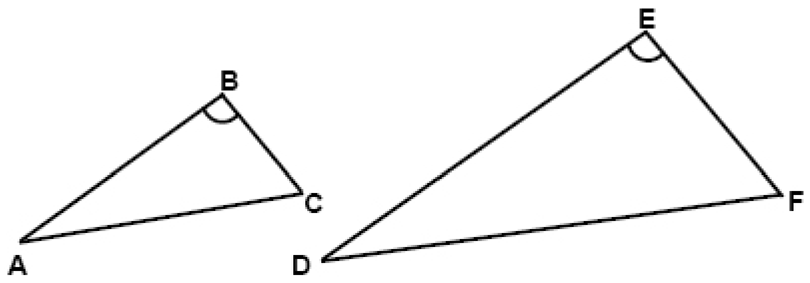
- The dimension of a space or object is informally defined as the minimum number of coordinates needed to specify each point within it.
- A line has a dimension of one because only one coordinate is needed to specify a point on it.
- A surface such as a plane or the surface of a cylinder or sphere has a dimension of two because two coordinates are needed to specify a point on it.
- The inside of a cube, a cylinder or a sphere is three-dimensional because three coordinates are needed to locate a point within these spaces.
- As one would expect, the (topological) dimension is always a natural number.

Inductive dimension

- Consider a discrete set of points (such as a finite collection of points) to be 0-dimensional.
- By dragging a 0-dimensional object in some direction, one obtains a 1-dimensional object.
- By dragging a 1-dimensional object in a new direction, one obtains a 2-dimensional object.
- In general, one obtains an $(n + 1)$-dimensional object by dragging an n-dimensional object in a *new direction*.
- The inductive dimension of a topological space may refer to the small inductive dimension or the large inductive dimension and is based on the analogy that $(n + 1)$-dimensional balls have n-dimensional boundaries, permitting an inductive definition based on the dimension of the boundaries of open sets.

Similar triangles

- In geometry two triangles, $\triangle ABC$ and $\triangle DEF$, are similar if and only if corresponding angles have the same measure: this implies that they are similar if and only if the lengths of corresponding sides are proportional.
- Two geometrical objects are called similar if one can be obtained from the other by uniformly scaling (enlarging or reducing), possibly with additional translation, rotation and reflection.



Metric space

A non-empty set V becomes a metric space when supplied with a mapping (metric) of the form $\rho\colon V \times V \to \mathbb{R}$ given by the formula $(x, y) \mapsto \rho(x, y)$ which for each $x, y, z \in V$ has the properties:

(M1) $\rho(x, y) \geq 0$, (non-negativity)

and

$\rho(x, y) = 0 \Leftrightarrow x = y$ (identity)

(M2) $\rho(x, y) = \rho(y, x)$ (symmetry)

(M3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (triangle inequality)

The members of V are frequently called 'points' and the non-negative, real number $\rho(x, y)$ the 'distance' from the 'point' x to the 'point' y.

Examples

- The set $\mathbb{R}$ of all real numbers with the usual metric $\rho(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ is a metric space, which is called a *real line*.
- The most important space for us is the familiar n-dimensional Euclidean space $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$ with the *Pythagorean* or *root mean square error metric* defined by

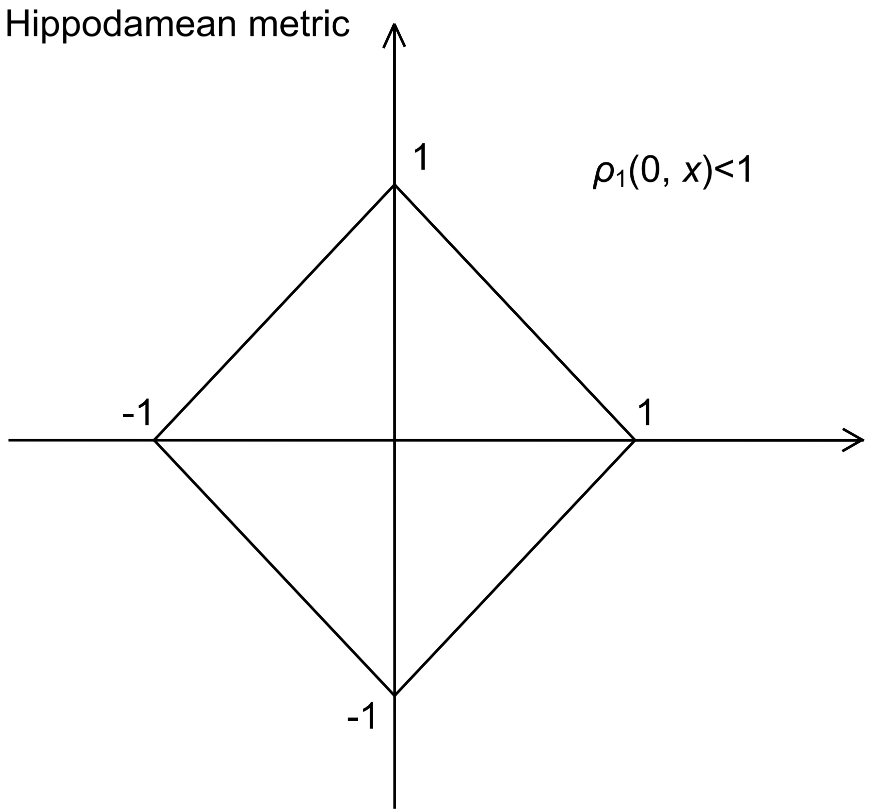
$$\rho_2(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$$

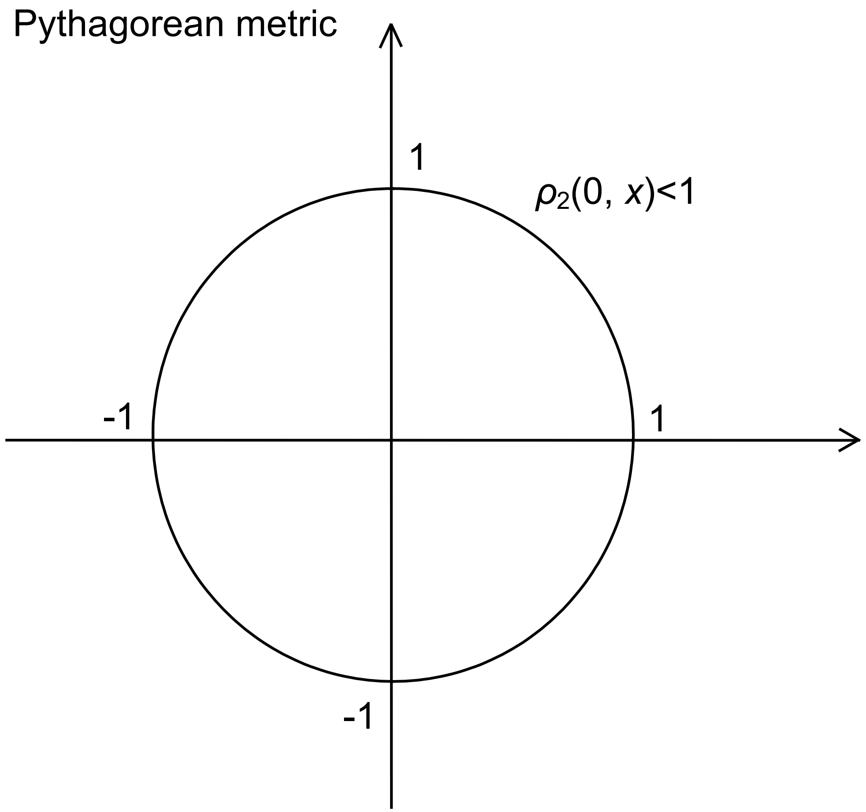
or with the *Hippodamean metric*

$$\rho_1(x, y) = \sum_i |x_i - y_i|$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $x_i, y_i \in \mathbb{R}$, sometimes called the *box* or *city-block* metric.

The locus


Hippodamean metric
1
$\rho_1(0, x)<1$
-1
1
-1


Pythagorean metric
1
$\rho_2(0, x)<1$
-1
1
-1

Similarity

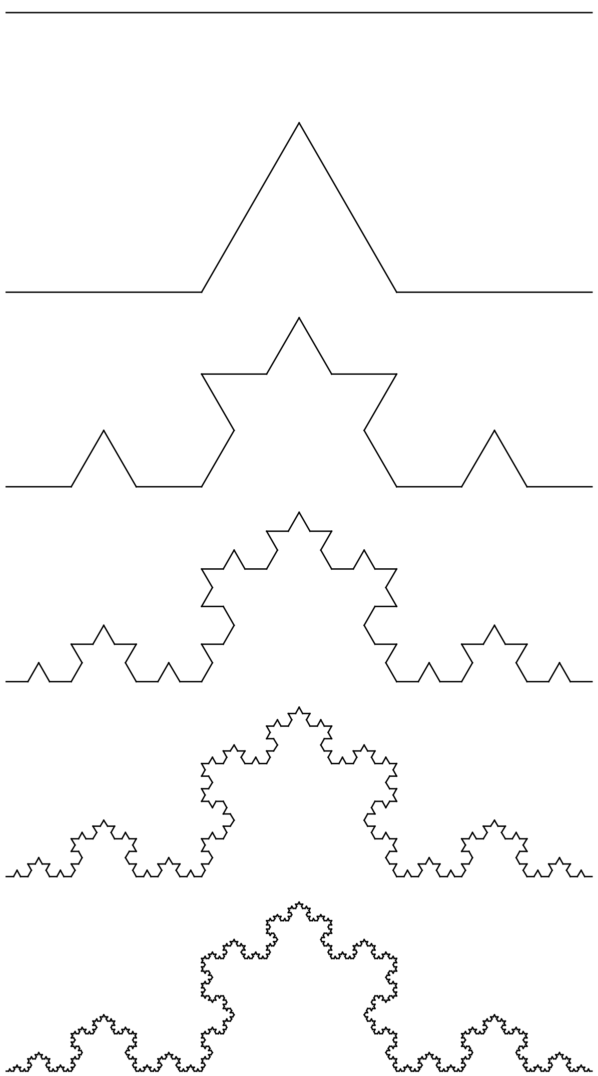
- A mapping $f\colon X \to Y$, where (X, ρ) and (Y, σ) are metric spaces is a similarity or similitude of ratio or scale r, if
$$\sigma(f(x), f(y)) = r\,\rho(x, y)$$
for every $x, y \in X$ and some fixed $r \in \mathbb{R}_+$.
- If the similarity ratio is greater than one, we have a *dilation*, whereas if the similarity ratio is less than one, we have a *contraction*.
- When $r = 1$ a similarity is called an isometry (rigid motion).

Self-similarity

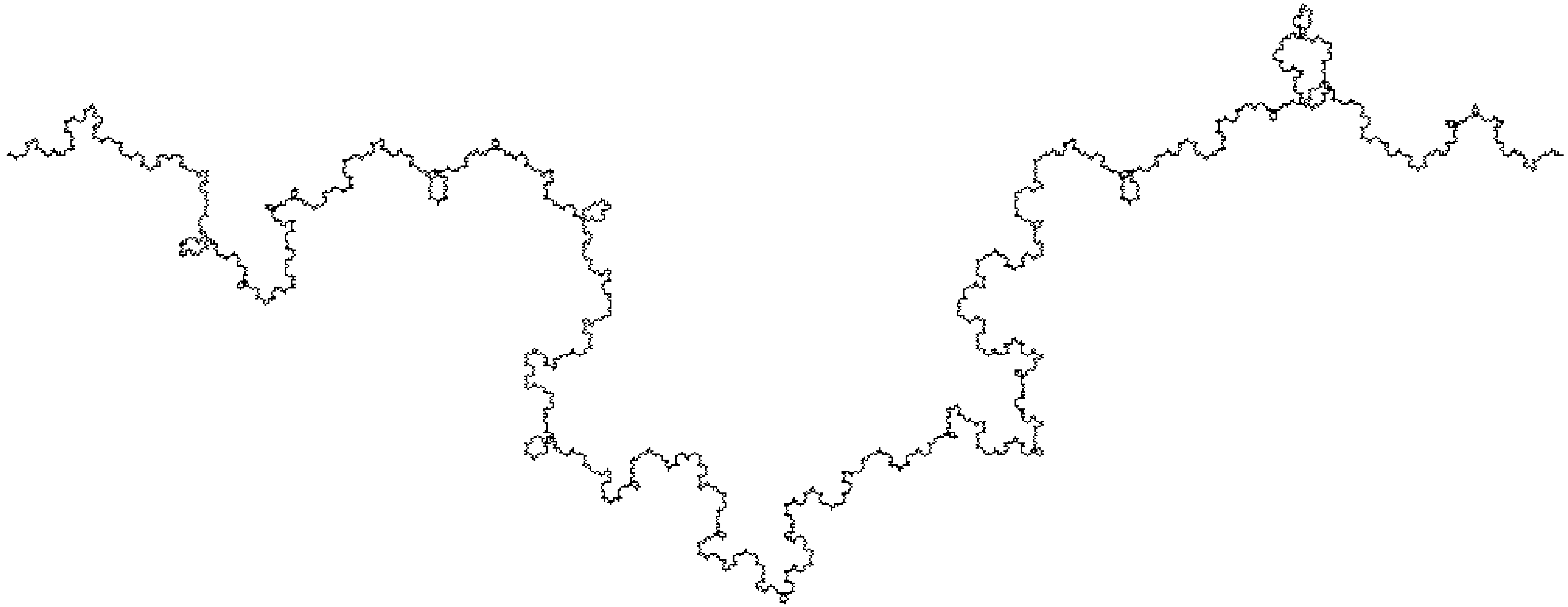
- A self-similar object is exactly or approximately similar to a part of itself (i.e. the whole has the same shape as one or more of the parts).
- Many objects in the real world, such as coastlines, are statistically self-similar: parts of them show the same statistical properties at many scales.
- Self-similarity is a typical property of fractals.
- Scale invariance is an exact form of self-similarity where at any magnification there is a smaller piece of the object that is similar to the whole.

Koch curve

- The single line segment in Step 0 is broken into four equal-length segments in Step 1.
- This same “rule” is applied an infinite number of times resulting in a figure with an infinite perimeter.
- The first five stages are shown on the right.



Randomly placed generator



Dimension 1.2619...

- Consider a Koch curve, where each of the 4 new lines is 1/3 the length of the old line.
- Blowing up the Koch curve by a factor of 3 results in a curve 4 times as large (one of the old curves can be placed on each of the 4 segments)
- Therefore, $4 = 3^d$ or

$$d = \frac{\ln 4}{\ln 3}.$$

Similarity dimension

- A set F is called self-similar, if
$$F = w_1(F) \cup w_2(F) \cup \cdots \cup w_N(F),$$
where w_i are similitudes with common similarity ratio r and the sets $w_i(F)$ do not overlap.
- For a self-similar shape F made of N copies of itself, each scaled by a similarity with contraction factor r, the similarity dimension is
$$\dim_s F = \frac{\log(N)}{\log(1/r)}.$$

Examples

- The graph of Weierstrass function $2 + \frac{\log a}{\log b}$ under constraints
- Cantor set $N = 2$, $r = 1/3$, $\dim_s \mathcal{C} = \log 2/\log 3 = 0{,}630929...$
- Koch snowflake $N = 4$, $r = 1/3$, $\dim_s K = \log 4/\log 3 = 1{,}261859507...$
- Sierpiński gasket $N = 3$, $r = 1/2$, $\dim_s S = \log 3/\log 2 = 1{,}584962500...$
- Sierpiński carpet $N = 8$, $r = 1/3$, $\dim_s C = \log 8/\log 3 = 1{,}892789260...$
- Peano curve $N = 9$, $r = 1/3$, $\dim_s P = \log 9/\log 3 = 2$
- Hilbert curve $N = 4$, $r = 1/2$, $\dim_s H = \log 4/\log 2 = 2$
- Moore curve $N = 4$, $r = 1/2$, $\dim_s M = \log 4/\log 2 = 2$
- Sierpiński tetrahedron $N = 4$, $r = 1/2$, $\dim_s T = \log 4/\log 2 = 2$
- Menger sponge $N = 20$, $r = 1/3$, $\dim_s M = \log 20/\log 3 = 2{,}726833028...$
- 3D Hilbert curve $N = 8$, $r = 1/2$, $\dim_s H = \log 8/\log 2 = 3$
- 3D Moore curve $N = 8$, $r = 1/2$, $\dim_s M = \log 8/\log 2 = 3$

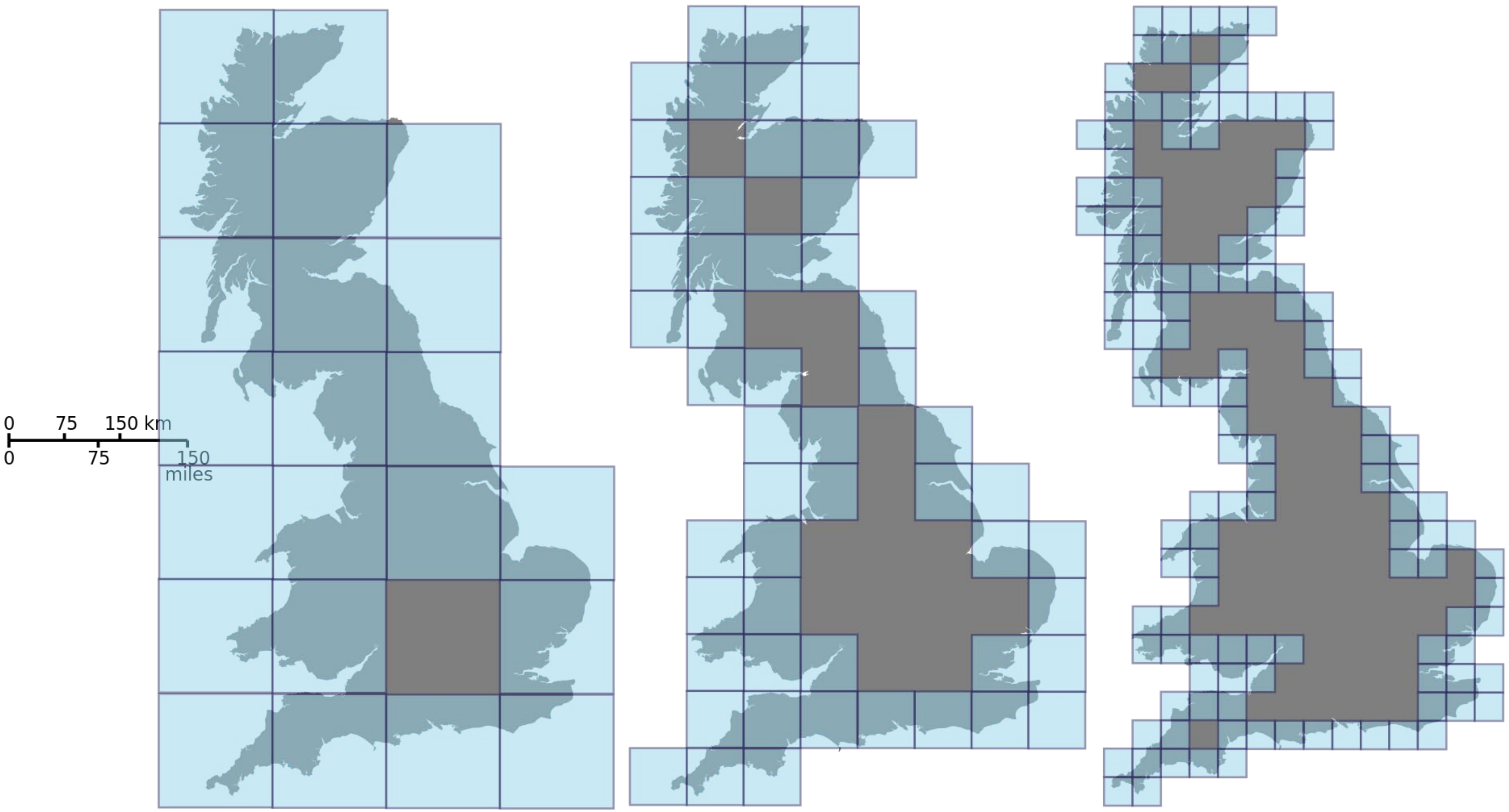
Box-counting dimension

- Let A be a set in a metric space.
- For each $\varepsilon > 0$, let $N(A, \varepsilon)$ denote the smallest number of closed balls of radius $\varepsilon > 0$ needed to cover A.
- If

$$D = \lim_{\varepsilon \to 0} \left\{ \frac{\ln(1/N(A,\varepsilon))}{\ln(\varepsilon)} \right\}$$

exists, then D is the box-counting dimension of A.

Estimating the box-counting dimension of the coast of Great Britain



Higuchi dimension

- It is an approximate value for the box-counting dimension of the graph of a real-valued function or time series.
- This value is obtained via an algorithmic approximation so one also talks about the Higuchi method.
- It has many applications in science and engineering and has been applied to subjects like characterising primary waves in seismograms, clinical neurophysiology and analysing changes in the electroencephalogram in Alzheimer's disease.

Hausdorff-Besicovitch dimension

- Let X be a metric space. If $S \subset X$ and $d \in [0, +\infty)$, the d-dimensional Hausdorff content of S is defined by

$$C_H^d(S) = \inf\left\{\sum_i r_i^d : \text{there is a cover of } S \text{ by balls with radii } r_i > 0\right\}.$$

- In other words, $C_H^d(S)$ is the infimum of the set of numbers $\delta \geq 0$ such that there is some (indexed) collection of balls $\{B(x_i, r_i) : i \in I\}$ covering S with $r_i > 0$ for each $i \in I$ which satisfies

$$\sum_{i \in I} r_i^d > \delta.$$

- The Hausdorff dimension of S is defined by

$$\dim_H(S) = \sup\left\{d \geq 0 : C_H^d(S) = \infty\right\} = \inf\left\{d \geq 0 : C_H^d(S) = 0\right\}.$$

Examples

- Let F be a flat disk of unit radius in $\mathbb{R}^3$.
- From familiar properties of length, area and volume $C_H^1(F)$= length (F) = ∞, 0 < $C_H^2(F)$= $(4/\pi)$ × area (F) = 4 < ∞ and $C_H^3(F)$ = $(6/\pi)$ × vol(F) = 0.
- Thus, $\dim_H F = 2$, with $C_H^d(F) = \infty$ if $d < 2$ and $C_H^d(F) = 0$ if $d > 2$.

Intuition

- The Hausdorff dimension measures the local size of a space taking into account the distance between points, the metric.
- Consider the number $N(r)$ of balls of radius at most r required to cover X completely.
- When r is very small, $N(r)$ grows polynomially with $1/r$. For a sufficiently well-behaved X, the Hausdorff dimension is the unique number d such that $N(r)$ grows as $1/r^d$ as r approaches zero.
- More precisely, this defines the box-counting dimension, which equals the Hausdorff dimension when the value d is a critical boundary between growth rates that are insufficient to cover the space, and growth rates that are overabundant.

Physical meaning

- Amount of variation in the object details
- A measure of roughness (fragmentation) of an object
- The concept was introduced in 1918 by the mathematician Felix Hausdorff.
- Many of the technical developments used to compute the Hausdorff dimension for highly irregular sets were obtained by Abram Samoilovitch Besicovitch.

What is a fractal?

- A fractal is by definition a set whose Hausdorff-Besicovitch dimension strictly exceeds its topological dimension.
- Since the dimension 1.2619 is greater than the dimension 1 of the lines making up the Koch curve, the curve is a fractal.

Outline

Outline

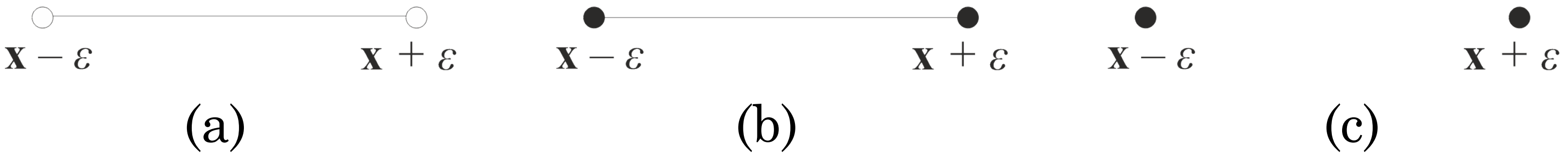
4. ITERATED FUNCTION SYSTEMS

- Preliminaries
- Distances between sets
- Dynamic systems

On circles and disks

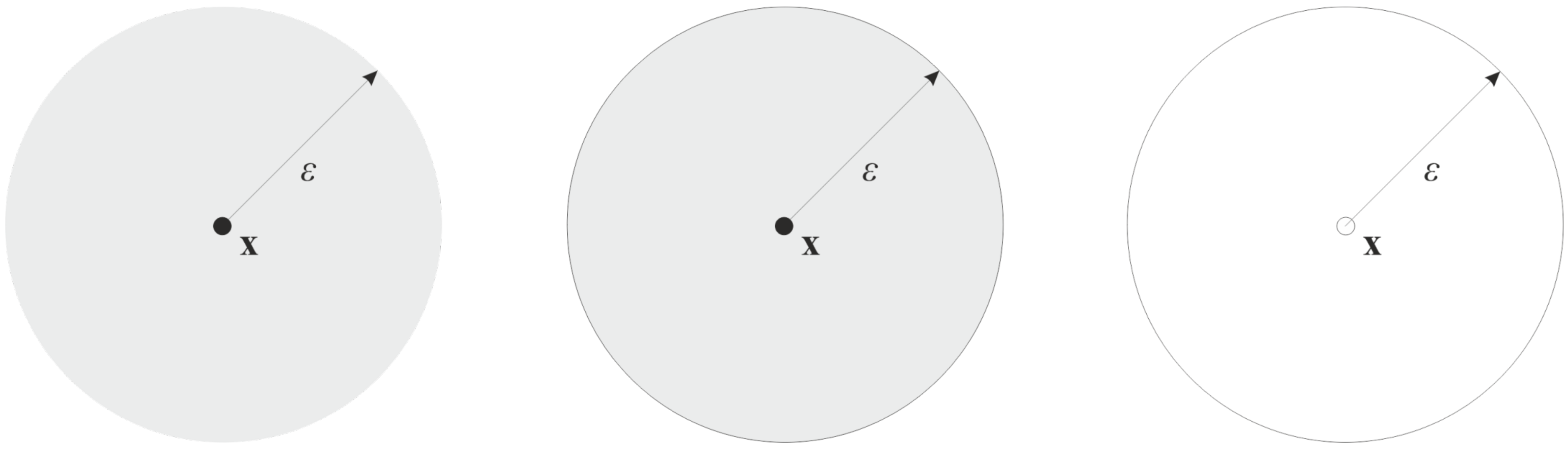
- The notion of distance in a metric space leads to the notion of the *open disk* which is the collection of points whose distance from the centre of the 'disk' equals a given positive real number.
- Hence, the set $D(x, \varepsilon) = \{ y \in X : \rho(x, y) < \varepsilon\}$ defines an open disk centred at $x \in X$ with radius $\varepsilon > 0$.
- An open disk is always an *open set* but *not all open sets are open disks*.
- An open set is the one which *contains an open disk for every point of the set*. In a dual sense, a *closed set* is the one whose complementary set is open.
- Hence, the set $\Delta(x, \varepsilon) = \{ y \in X : \rho(x, y) \leq \varepsilon\}$ defines a *closed disk* centred at $x \in X$ with radius $\varepsilon > 0$.
- The set $K(x, \varepsilon) = \{ y \in X : \rho(x, y) = \varepsilon\}$ defines a *circle* centred at $x \in X$ with radius $\varepsilon > 0$.

Example 1



(a) Open disk, (b) closed disk and (c) circle centred at $\mathbf{x}$ and of radius ε with respect to the usual metric of $\mathbb{R}$

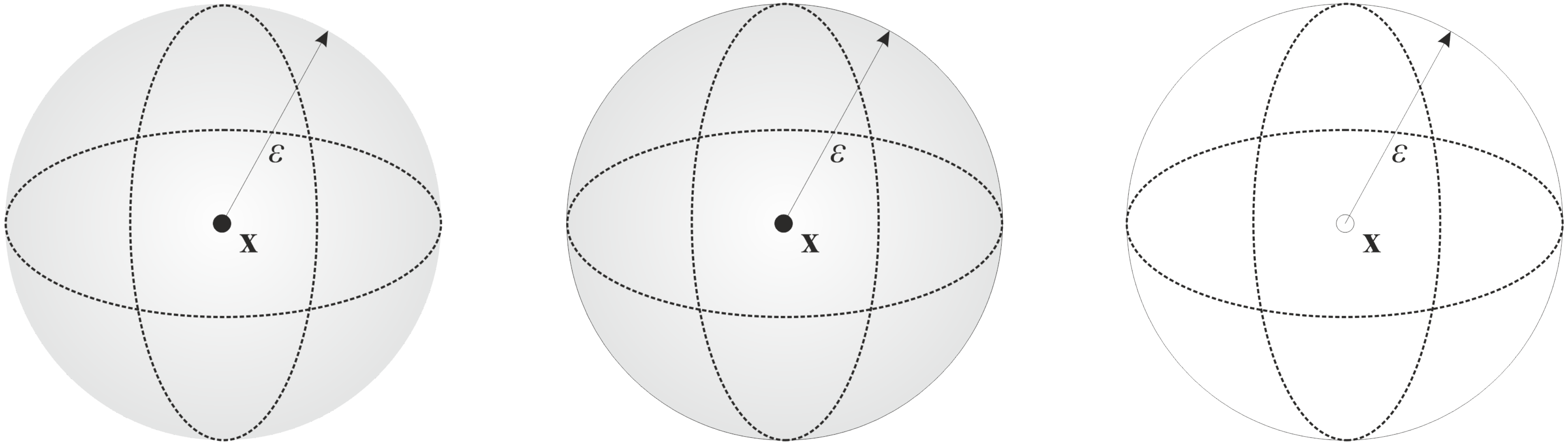
Example 2



(a) (b) (c)

(a) An open disk, (b) a closed disk and (c) a circle centred at **x** and of radius ε with respect to the ρ_2 metric of $\mathbb{R}^2$

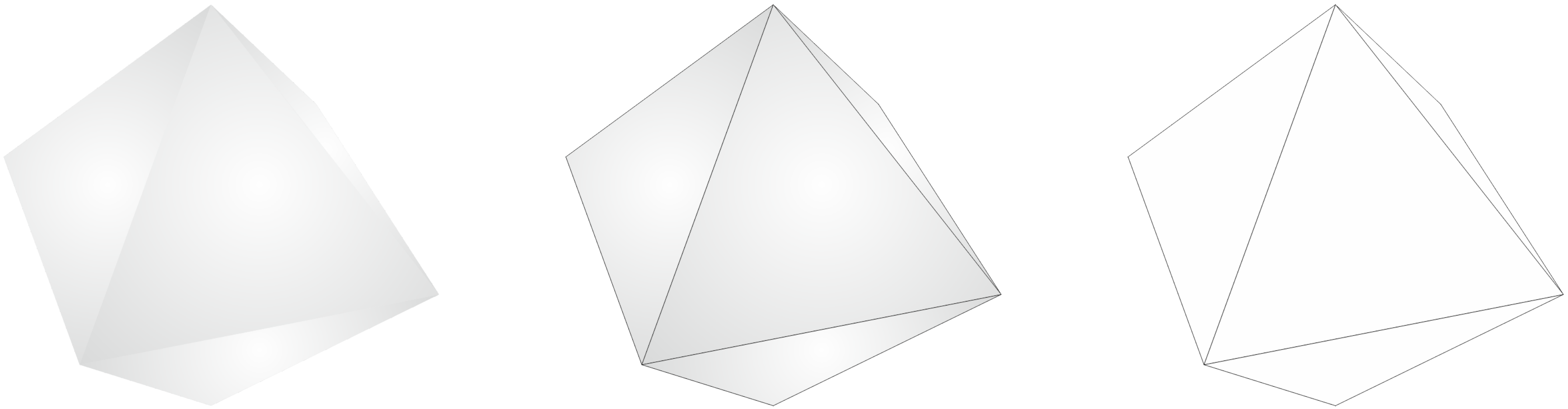
Example 3



(a) (b) (c)

(a) An open disk, (b) a closed disk and (c) a circle centred at **x** and of radius ε with respect to the ρ_2 metric of $\mathbb{R}^3$

Example 4



(a) (b) (c)

(a) An open disk, (b) a closed disk and (c) a circle centred at $\mathbf{x}$ and of radius ε with respect to the ρ_1 metric of $\mathbb{R}^3$

More topology

- The *closure* of a set $A \subset X$, denoted by $\bar{A}$, is the smallest closed set containing A.
- The set $A \subset X$ is called *bounded*, if there exists $M > 0$ and $x_0 \in X$ such that $A \subset D(x_0, M)$.
- A subset K of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

The space where fractals live

- Let (X, ρ) be a metric space. Then, $\mathcal{H}(X)$ denotes the space whose points are the compact subsets of X, other than the empty set, i.e.

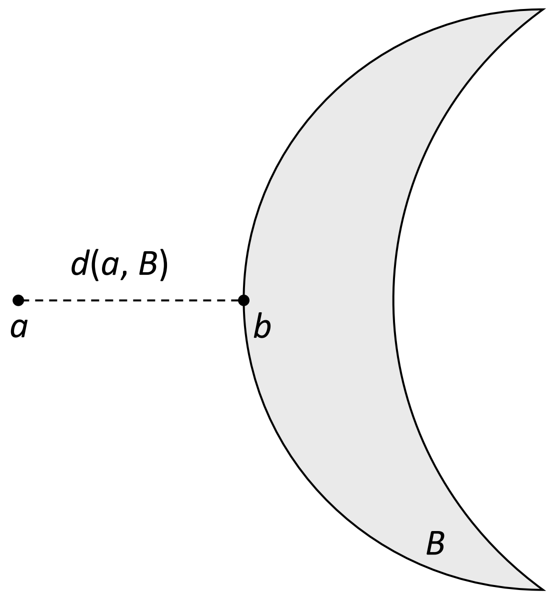
$$\mathcal{H}(X) = \{\emptyset \neq A \subset X : A \text{ is compact}\}.$$

- Sometimes $\mathcal{H}(X)$ is referred to as the 'space of fractals in X' (but note that not all members of $\mathcal{H}(X)$ are fractals).
- The difference between a subset of $\mathcal{H}(X)$ and a nonempty, compact subset of X is that $\mathcal{H}(X)$ is a set of sets, so every subset of it is a set of compact sets.

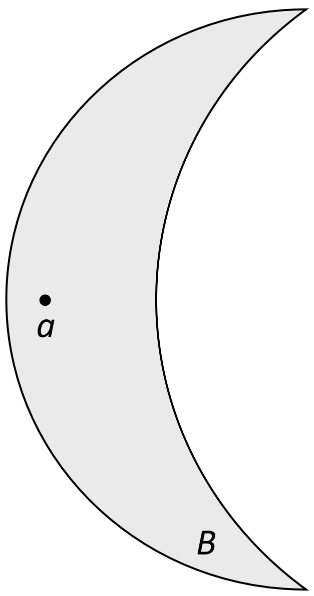
Distance between a point and a set

- The subset of real numbers $\{\rho(x, y) : y \in B\}$, where $x \in X$ and $B \in \mathcal{H}(X)$ has a smallest value.
- Then, as the distance of the point x from the subset B we consider

$$\min\{\rho(x, y) : y \in B\}.$$



$d(a, B) = \rho(a, b)$



$d(a, B) = 0$

Distances between sets

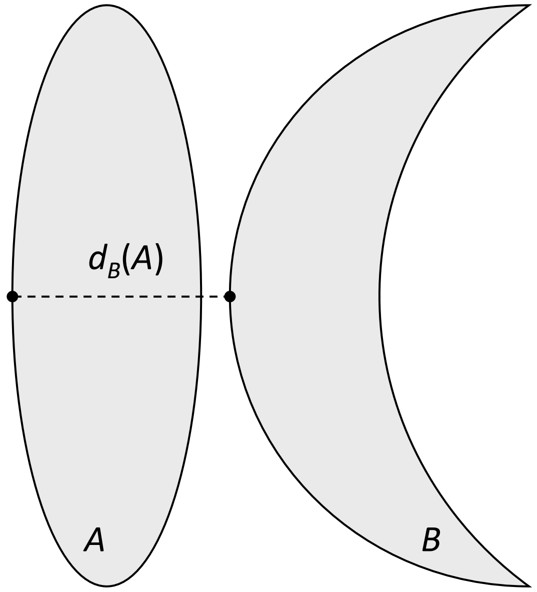
- Let A and B be two nonempty, compact subsets of a metric space (X, ρ). We define as

$$d_A(B) = \max\{d(x, A) : x \in B\}$$

and

$$d_B(A) = \max\{d(x, B) : x \in A\}.$$

- The function $d_B(A)$ is usually called the directed Hausdorff distance from A to B.

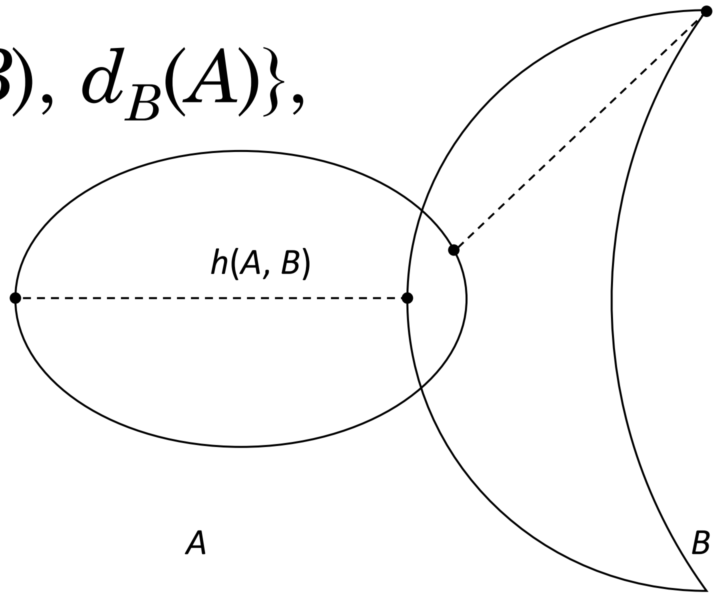


The Hausdorff metric

- It measures how far two subsets of a metric space are from each other.
- It turns the set of nonempty, compact subsets of a metric space into a metric space in its own right.
- If

$$h(A, B) = \max\{d_A(B), d_B(A)\},$$

then $(\mathcal{H}(X), h)$ is a metric space.



More topology

- One of the most useful properties a metric space may have, has to do with the so-called *Cauchy sequences*. We say that a sequence $\{x_n \mid n \in \mathbb{N}\}$ is *a Cauchy sequence* iff $\forall\ \varepsilon > 0,\ \exists\ n_0 \in \mathbb{N}$: $\rho(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.
- That is, a Cauchy sequence is *a sequence whose elements come closer and closer as n increases*. In general, *every converging sequence is always a Cauchy sequence.*
- Whenever the reverse holds, we say that our metric space is *complete*; more formally, a metric space is by definition *complete if and only if a Cauchy sequence is a converging sequence.*
- $(\mathcal{H}(X), h)$ is a complete metric space whenever (X, ρ) is a complete metric space.

Iterated function

- In mathematics, an *iterated function* is a function which is composed with itself, possibly ad infinitum, in a process called iteration.
- *Iteration* means the act of repeating a process with the aim of approaching a desired goal, target or result.
- The formal definition of an iterated function on a set X follows.

Dynamic system

- Define f^k as the k-th iterate of f, where k is a non-negative integer, by $f^0 = \mathrm{id}_X$ and $f^{k+1} = f \circ f^k$, where id_X is the identity function on X and $f \circ g$ denotes function composition.
- Let S be a subset of $\mathbb{R}^n$ and let f: $S \to S$ be a continuous mapping. An iterative scheme $\{f^k\}$ is called a discrete dynamic system.
- A periodic point of period n of the transformation f: $X \to X$ is a point $x \in X$ such that $f^n(x) = x$ for some $n \in \mathbb{N}$. The smallest positive integer n satisfying the above is called the *prime period* or *least period* of the point x.
- A periodic point of f of period 1 is called a fixed point of f.
- The orbit of a periodic point of f is called a cycle or periodic orbit of f.
- We are interested in the behaviour of the sequence of iterates, or orbits, $\{f^k(x)\}$ for various initial points $x \in S$, particularly for large k.

Some fixed-point theorems

- (Brouwer) *Every continuous function from a closed disk to itself has at least one fixed point*. The theorem holds only for functions that are endomorphisms (functions that have the same set as the domain and codomain) and for nonempty sets that are compact (thus, in particular, bounded and closed) and convex (or homeomorphic to convex).
- Let $f: X \to X$ be a continuous mapping, where (X, ρ) is a compact metric space. Then there exists a nonempty, closed set $A \subset X$ such that

$$f(A) = A.$$

Contraction mapping

- A contraction mapping, or contraction, on a metric space (X, ρ) is a function f from X to itself, with the property that there is a nonnegative real number $s < 1$ such that for all x and y in X,

$$\rho(f(x), f(y)) \leq s \cdot \rho(x, y).$$

- The smallest such value of s is called the Lipschitz constant of f.
- Contractive maps are sometimes called Lipschitzian maps.
- If the above condition is satisfied for $s \leq 1$, then the mapping is said to be non-expansive.

Banach fixed point theorem

- Also known as the contraction mapping theorem or contraction mapping principle.
- Let (X, ρ) be a nonempty, complete metric space. Let $T: X \to X$ be a contraction mapping on X.
- Then the map T admits one and only one fixed point x^* in X (this means $T(x^*) = x^*$).
- Furthermore, this fixed point can be found as follows: Start with an arbitrary element x_0 in X and define an iterative sequence by $x_n = T(x_{n-1})$ for $n = 1, 2, 3, \ldots$. This sequence converges and its limit is x^*.

The attractor

- We shall call a subset F of S an attractor for f if F is a closed set that is invariant under f (i.e., $f(F) = F$) such that the distance from $f^k(x)$ to F converges to zero as k tends to infinity for all x in an open set V containing F.
- The set V is called the basin of attraction of F.

Iterated Function Systems (IFS's)

A (hyperbolic) *Iterated Function System* (IFS) on the metric space $(\mathbb{R}^n, ||\cdot||)$ is defined as a pair $\{\mathbb{R}^n; w_{1-M}\}$, where

$$\{w_i : \mathbb{R}^n \to \mathbb{R}^n,\ i = 1, 2, \ldots, M\}$$

is a finite set of contractions with *contractivity factors* s_i, i.e. for every $i = 1, 2, \ldots, M$

$$||w_i(\mathbf{x}) - w_i(\mathbf{y})|| \leq s_i\ ||\mathbf{x} - \mathbf{y}||\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $0 \leq s_i < 1$.

Hutchinson operator

- A collection of functions on an underlying space X.
- Formally, let $\{\mathbb{R}^n; w_{1-M}\}$ be an IFS, or a set of M contractions from a compact set X into itself. We may regard this as defining an operator H on the power set 2^X as

$$H : A \mapsto \bigcup_{i=1}^{M} w_i(A),$$

where A is any subset of X.
- The iteration of these functions gives rise to the attractor of an iterated function system, for which the fixed set is self-similar.

The attractor of an IFS

- The attractor of a (hyperbolic) IFS is the unique set

$$\mathcal{A}_\infty = \lim_{k\to\infty} H^k(A_0)$$

for every starting set A_0, where

$$H(A) = \bigcup_{i=1}^{M} w_i(A)$$

for all $A \in \mathcal{H}(\mathbb{R}^n)$.

- The map H is also called the collage map to alert us to the fact that $H(A)$ is formed as a union or 'collage' of sets.

Affine transformations

A transformation w is affine, if it may be represented by a matrix $\mathbf{A}$ and translation $\mathbf{t}$ as $\mathbf{w(x) = Ax + t}$, or, if $X = \mathbb{R}^2$,

$$w\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d \\ e \end{pmatrix}$$

whereas if $X = \mathbb{R}^3$

$$w\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & g \\ h & k & s \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} l \\ m \\ r \end{pmatrix}.$$

Example: Modified Sierpiński gasket

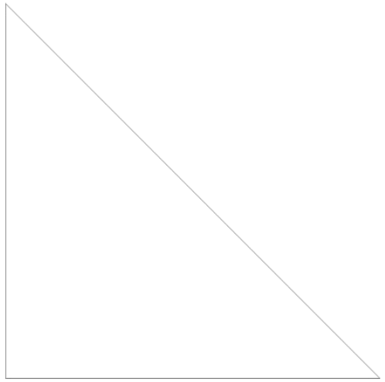
Consider an IFS of the form $\{\mathbb{R}^2; w_1, w_2, w_3\}$, where

$$w_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

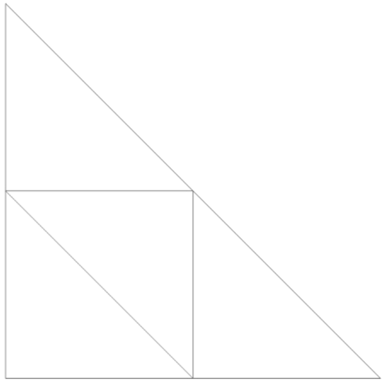
$$w_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$w_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

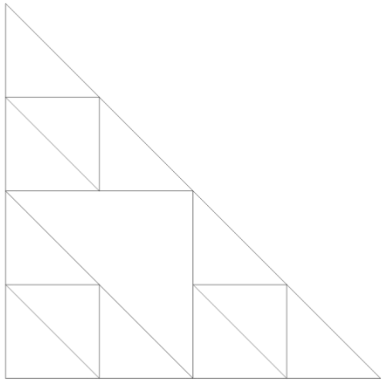
Photocopy machine 1



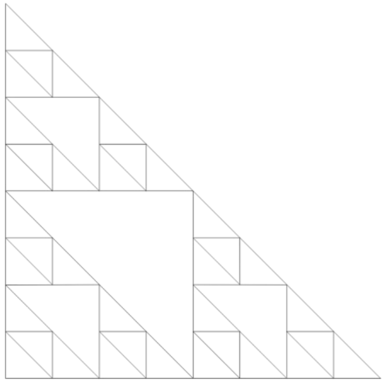
A_0



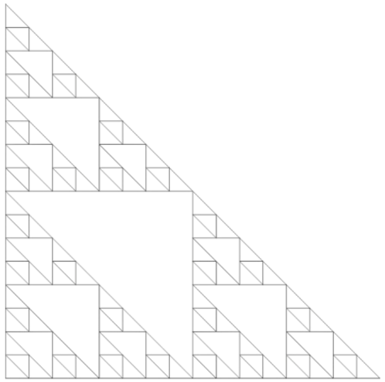
$H(A_0)$



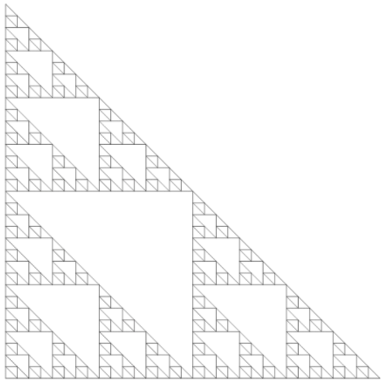
$H^2(A_0)$



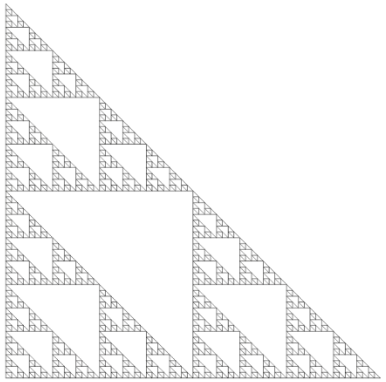
$H^3(A_0)$



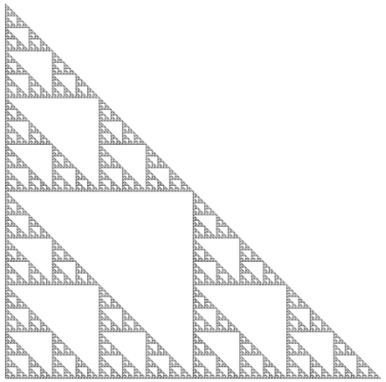
$H^4(A_0)$



$H^5(A_0)$

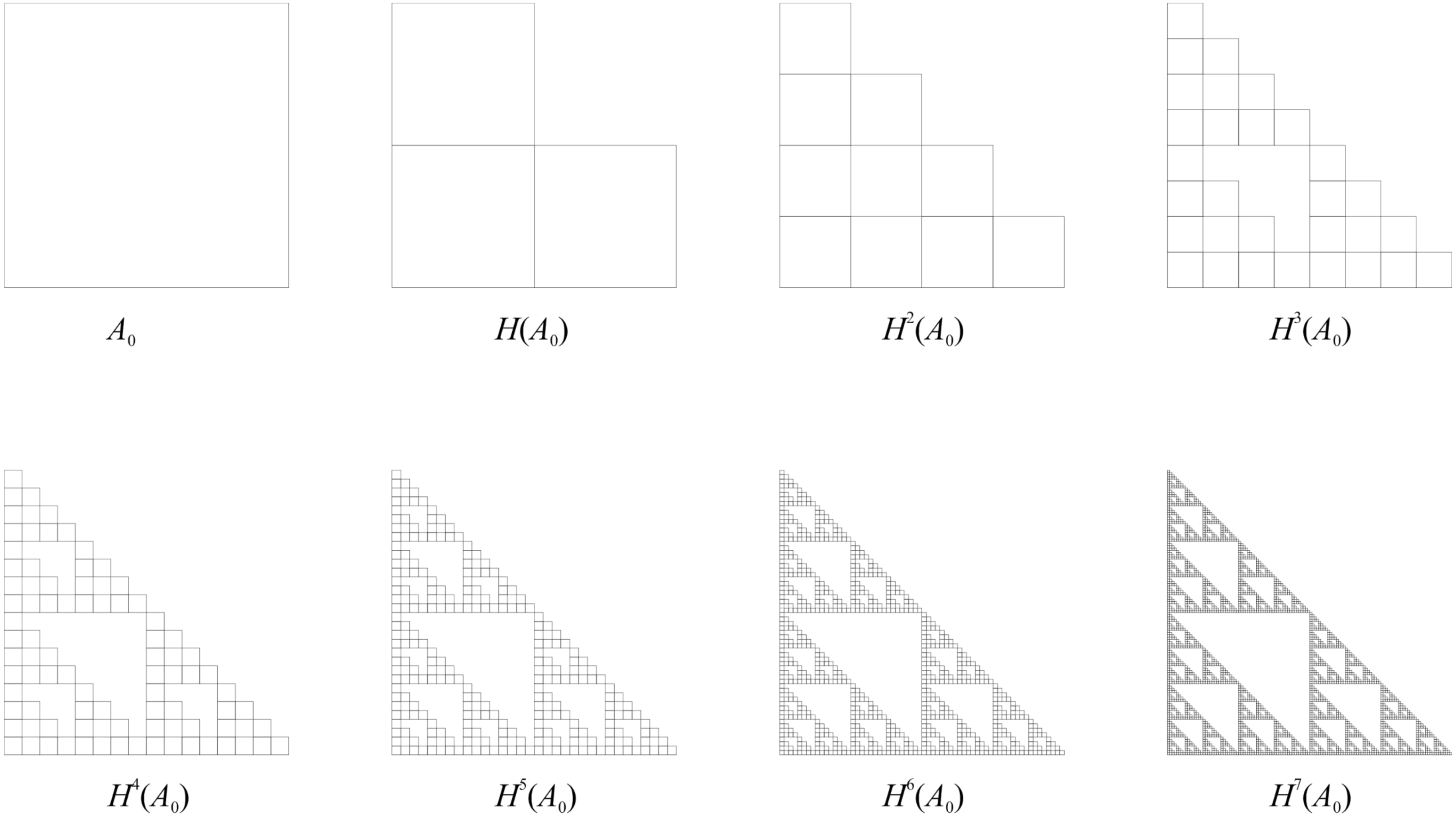


$H^6(A_0)$

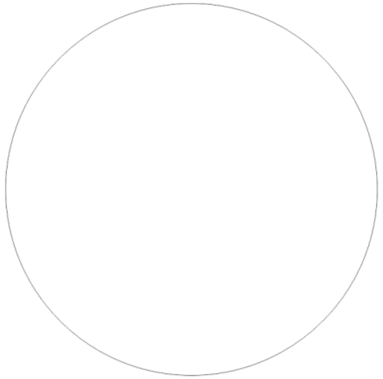


$H^7(A_0)$

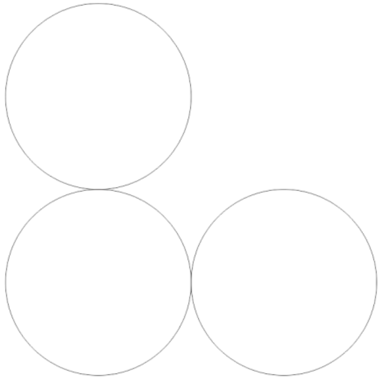
Photocopy machine 2



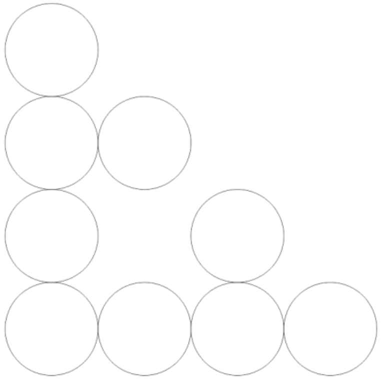
Photocopy machine 3



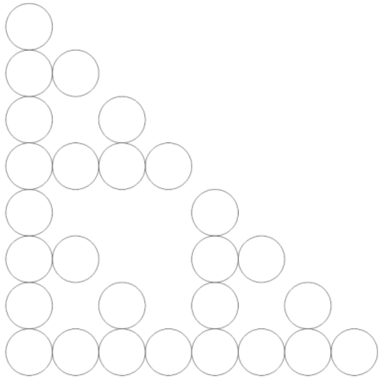
A_0



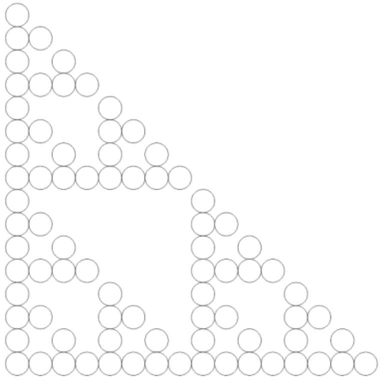
$H(A_0)$



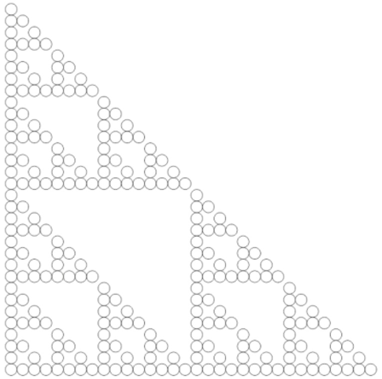
$H^2(A_0)$



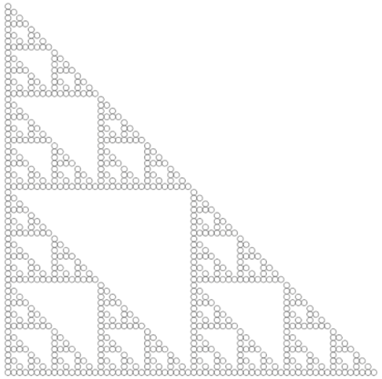
$H^3(A_0)$



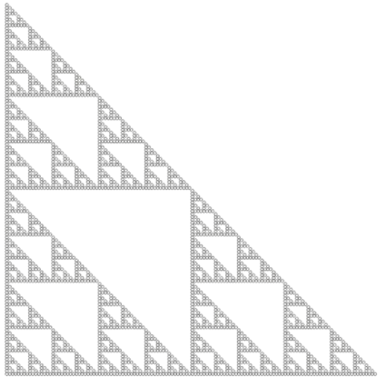
$H^4(A_0)$



$H^5(A_0)$

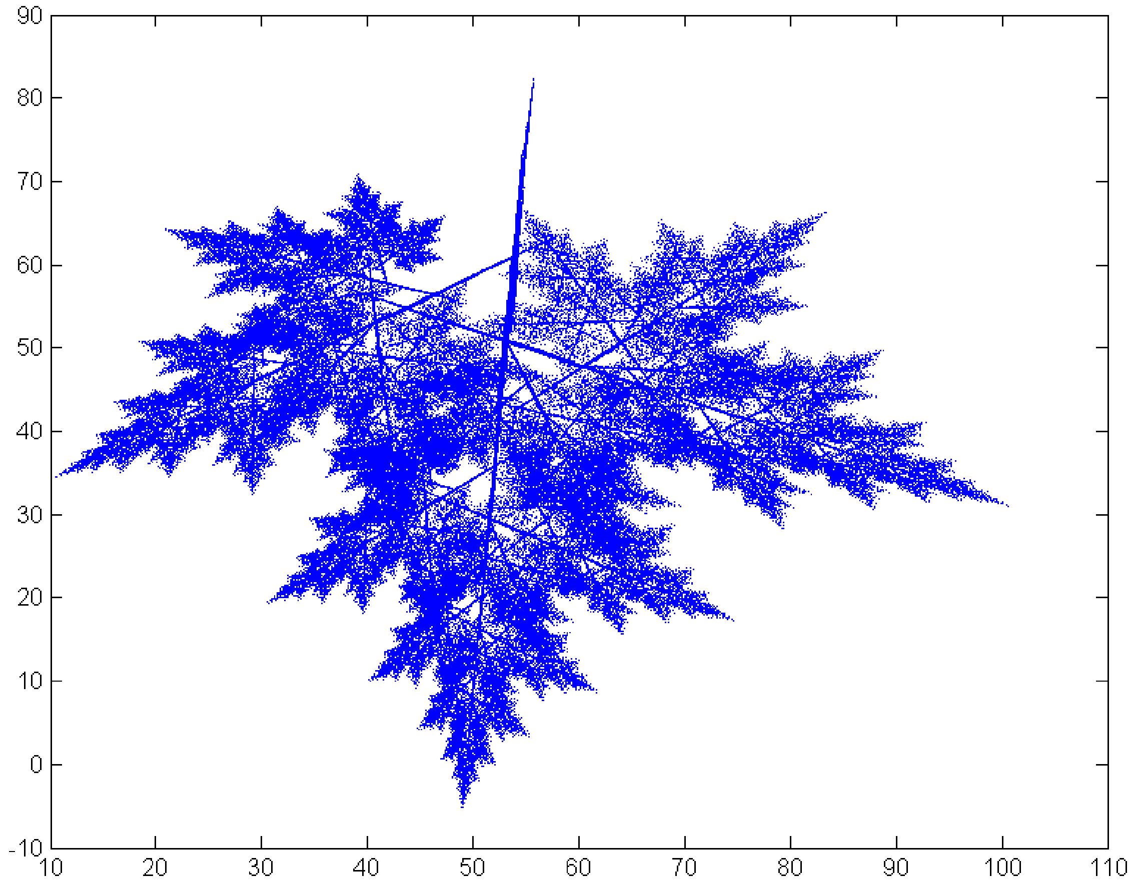


$H^6(A_0)$

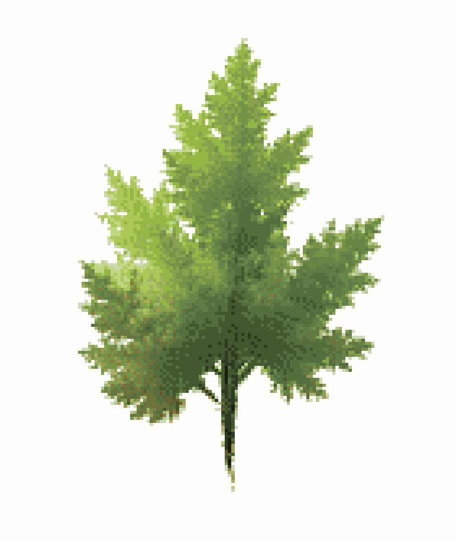


$H^7(A_0)$

Complexity



The geometry of nature



Recurrent IFSs

- An IFS with probabilities, written formally as $\{X; w_1, w_2, \ldots, w_M; p_1, p_2, \ldots, p_M\}$ or, somewhat more briefly, as $\{X; w_{1-M}; p_{1-M}\}$, gives to each transformation in H a probability or weight.
- If the weights of transformations differ, so do the measures on different parts of the attractor.
- A non-self-similar attractor, however, is more easily represented with a recurrent iterated function system, or RIFS for short.
- Each transformation has, instead of a single weight for the next iteration, a vector of weights for each transformation, $\{X; w_{1-M}; p_{i,j} \in [0, 1]; i, j = 1, 2, \ldots, M\}$, so that the matrix of weights is a recurrent Markov operator for the Hutchinson operator's transformation.

Outline

Outline

5. FRACTAL INTERPOLATION

- Introduction
- Functions
- Surfaces

Why interpolation functions?

- Euclidean geometry and elementary functions are the basis of the traditional methods for analyzing experimental data
- These functions can be expressed by simple mathematical formulas
- They can be stored in small files and computed by fast algorithms

Why fractal?

- Integral dimension
- Suitable for the design of man-made objects (e.g. circles, squares)

- Non-integral dimension
- Suitable for the design of natural objects (e.g. clouds, mountain ranges)
- Better fitting to experimental data (e.g. EEG, ECG, seismograph, image compression)

An example…

A fractal interpolation function.

Interpolation functions in $\mathbb{R}$

- Let the continuous function f be defined on a real closed interval $I = [x_0,x_M]$ and with range the metric space $(\mathbb{R}, |\cdot|)$, where
$$x_0 < x_1 < \cdots < x_M.$$
It is not assumed that these points are equidistant.
- The function f is called an interpolation function corresponding to the generalised set of data
$$\{(x_m, y_m) \in K = I{\times}\mathbb{R} : m = 0,1,\ldots,M\},$$
if $f(x_m) = y_m$ for all $m = 0,1,\ldots,M$ and $K = I{\times}\mathbb{R}$.
- The points $(x_m, y_m) \in \mathbb{R}^2$ are called the interpolation points. We say that the function f *interpolates* the data and that (the graph of) f passes through the interpolation points.

Affine fractal interpolation

Let us represent our, real valued, set of data points as

$$\{(u_n, v_n) : n = 0, 1, \ldots, N;\ u_n < u_{n+1}\}$$

and the interpolation points as

$$\{(x_m, y_m) : m = 0, 1, \ldots, M;\ M \leq N\},$$

where u_n is the sampled index and v_n the value of the given point in u_n.

The affine fractal interpolation function (AFIF) is constructed with M affine mappings of the form

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_i \\ e_i \end{pmatrix}$$

where $s_i \in (-1,1)$ is the (free) vertical scaling factor, whereas the coefficients a_i, c_i, d_i, e_i arise from the constraints

$$w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} \quad \text{and} \quad w_i \begin{pmatrix} x_M \\ y_M \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad i = 1, 2, \ldots, M.$$

Affine fractal interpolation

- Solving the above equations results in

$$a_i = \frac{x_i - x_{i-1}}{x_M - x_0}, \qquad d_i = \frac{x_M x_{i-1} - x_0 x_i}{x_M - x_0}$$

$$c_i = \frac{y_i - y_{i-1}}{x_M - x_0} - s_i \frac{y_M - y_0}{x_M - x_0}, \qquad e_i = \frac{x_M y_{i-1} - x_0 y_i}{x_M - x_0} - s_i \frac{x_M y_0 - x_0 y_M}{x_M - x_0}$$

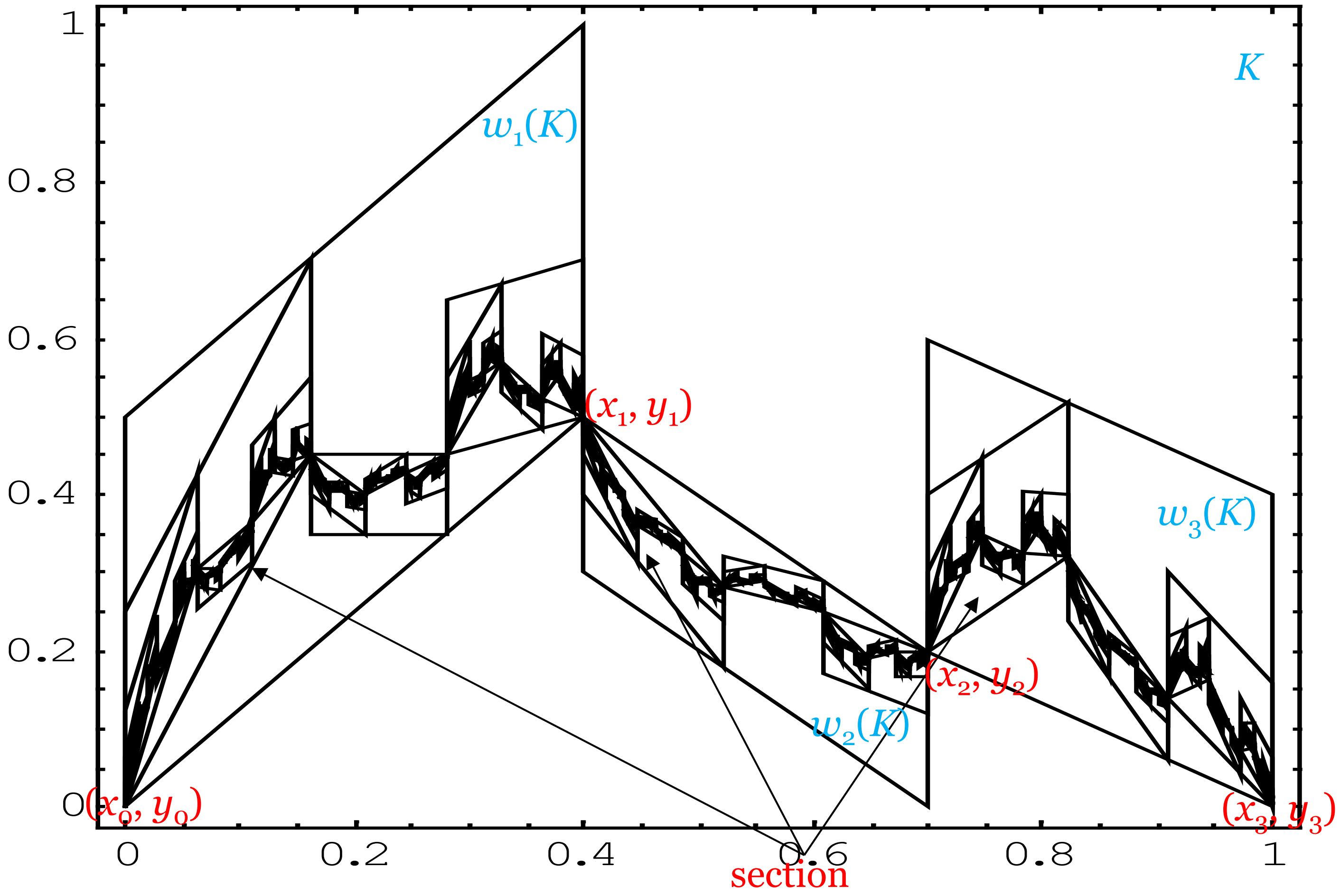
i.e. the coefficients a_i, c_i, d_i, e_i are completely determined by the interpolation points, while the s_i are free parameters satisfying $|s_i| < 1$ in order to guarantee that the IFS is hyperbolic with respect to an appropriate metric for every $i = 1, 2, \ldots, M$.

- The transformations w_i are shear transformations: line segments parallel to the y-axis are mapped to line segments parallel to the y-axis contracted by the factor $|s_i|$.
- For this reason, the s_i is called *vertical scaling* (or *contractivity*) *factor*.

IFS and interpolation functions

- The IFS $\{\mathbb{R}^2; w_{1-M}\}$ has a unique attractor, that is the graph of some continuous function which interpolates the data points.
- This function is called a fractal interpolation function (FIF), because its graph usually has non-integral dimension.
- A *section* is defined as the function values between interpolation points. It is a function with a self-affine graph since each affine transformation w_i maps the entire (graph of the) function to its section. The above function is known as affine FIF, or AFIF for short.

1D fractal interpolation



We map the entire (graph of the) function to each section of it.

Piecewise affine fractal interpolation

A pair of data points, which are called addresses, is now associated with each w_i

$$\{(\tilde{x}_{i,j}, \tilde{y}_{i,j}) : i = 1, 2, \ldots, M;\ j = 1, 2\}.$$

The domain is now the pair of addresses.

The constraints of the above mentioned case become

$$w_i \begin{pmatrix} \tilde{x}_{i,1} \\ \tilde{y}_{i,1} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} \quad \text{and} \quad w_i \begin{pmatrix} \tilde{x}_{i,2} \\ \tilde{y}_{i,2} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

subjected to

$$\tilde{x}_{i,2} - \tilde{x}_{i,1} > x_i - x_{i-1} \qquad i = 1, 2, \ldots, M.$$

Affine fractal interpolation

- Solving the above equations results in

$$a_i = \frac{x_i - x_{i-1}}{\tilde{x}_{i,2} - \tilde{x}_{i,1}} \qquad d_i = \frac{\tilde{x}_{i,2}x_{i-1} - \tilde{x}_{i,1}x_i}{\tilde{x}_{i,2} - \tilde{x}_{i,1}}$$

$$c_i = \frac{y_i - y_{i-1}}{\tilde{x}_{i,2} - \tilde{x}_{i,1}} - s_i \frac{\tilde{y}_{i,2} - \tilde{y}_{i,1}}{\tilde{x}_{i,2} - \tilde{x}_{i,1}} \qquad e_i = \frac{\tilde{x}_{i,2}y_{i-1} - \tilde{x}_{i,1}y_i}{\tilde{x}_{i,2} - \tilde{x}_{i,1}} - s_i \frac{\tilde{x}_{i,2}\tilde{y}_{i,1} - \tilde{x}_{i,1}\tilde{y}_{i,2}}{\tilde{x}_{i,2} - \tilde{x}_{i,1}}$$

  for every $i = 1, 2, \ldots, M$.
- The function constructed as the attractor of the above-mentioned IFS is called *recurrent fractal interpolation function*, or RFIF shortly, corresponding to the interpolation points.
- A RFIF is a piecewise self-affine function since each affine transformation w_i maps the part of the (graph of the) function defined by the corresponding address interval to each section.

Constraints

- For practical reasons suppose that the distance between the interpolation points along the horizontal and vertical direction is δ.
- We mapped the entire (graph of the) function to each section of the function. Now we map domains of the function to sections of the function. Suppose that each domain has size Δ.
- Points within a given interpolation section are not necessarily contained within any domain.

Interpolation functions in $\mathbb{R}^2$

- Let the discrete data

$$\{(x_i, y_j, z_{ij} = z(x_i, y_j)) \in \mathbb{R}^3 : i = 0, 1, \ldots, N; j = 0, 1, \ldots, M\}$$

  be known.
- Each affine mapping that comprises the hyperbolic IFS $\{\mathbb{R}^3; w_{1-N,\,1-M}\}$ is given by

$$w_{nm}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{nm} & b_{nm} & 0 \\ c_{nm} & d_{nm} & 0 \\ e_{nm} & g_{nm} & s_{nm} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} h_{nm} \\ k_{nm} \\ l_{nm} \end{pmatrix},$$

  with $|s_{nm}| < 1$ for every $n = 1, 2, \ldots, N$ and $m = 1, 2, \ldots, M$. The condition

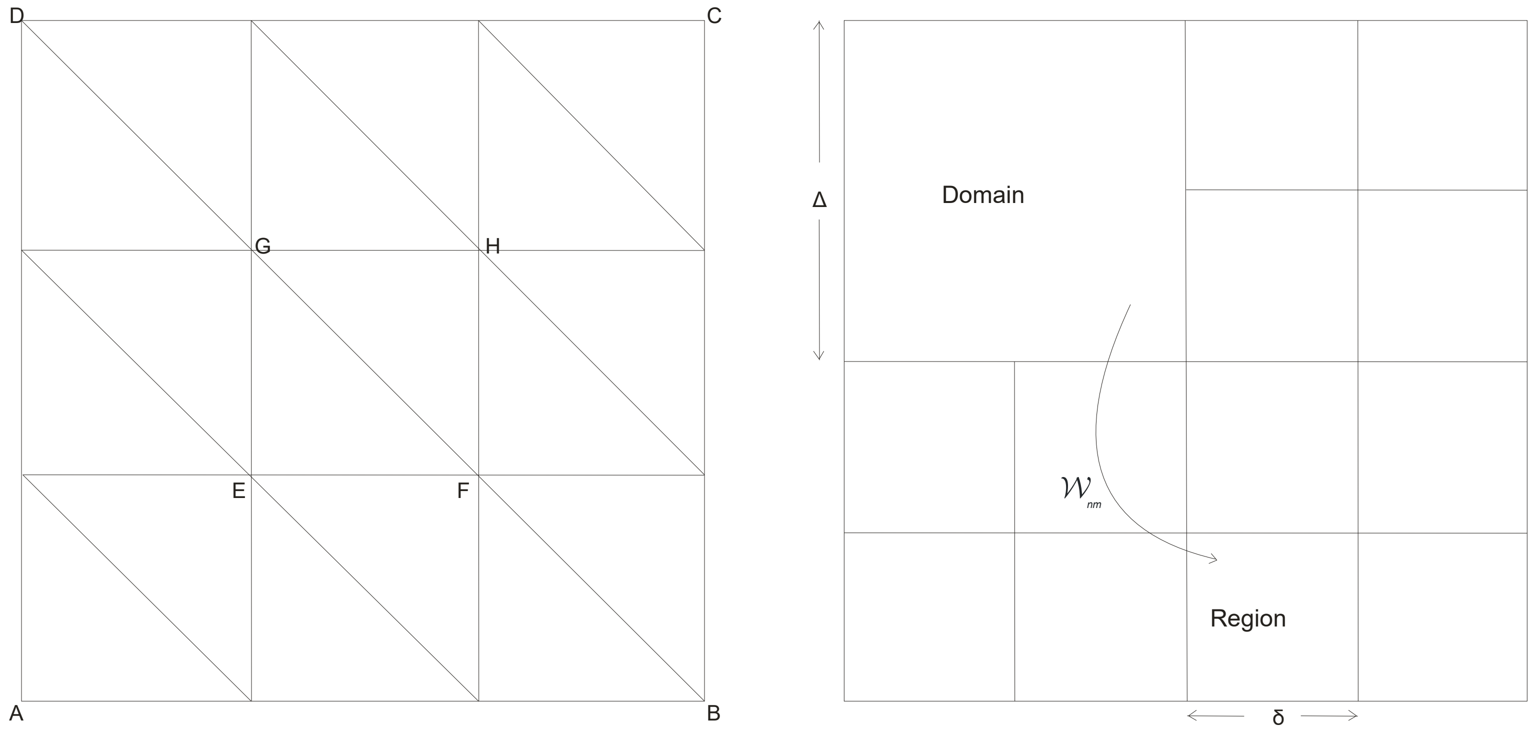
$$\left\| \begin{matrix} a_{nm} & b_{nm} \\ c_{nm} & d_{nm} \end{matrix} \right\| < 1$$

  ensures that

$$u_{nm}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{nm} & b_{nm} \\ c_{nm} & d_{nm} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_{nm} \\ k_{nm} \end{pmatrix}$$

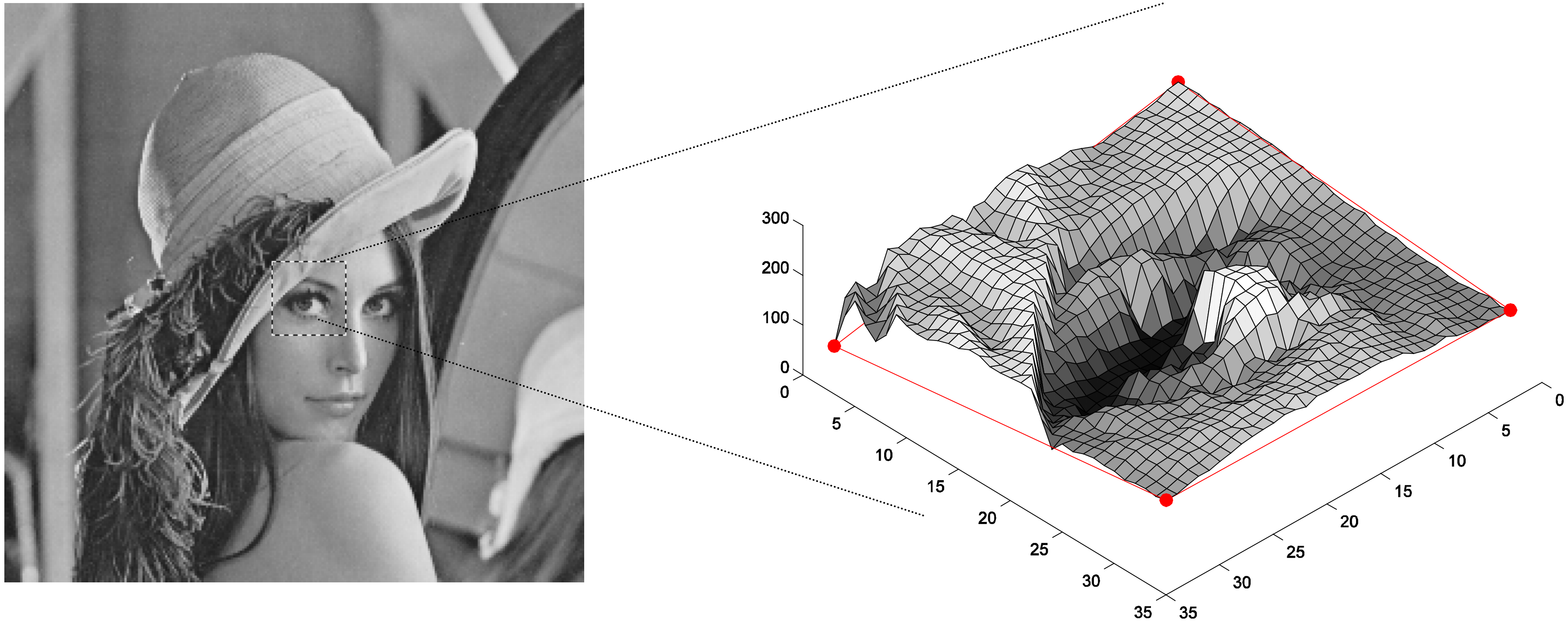
  is a similitude and the transformed surface does not vanish or flip over.

Rectangular lattices

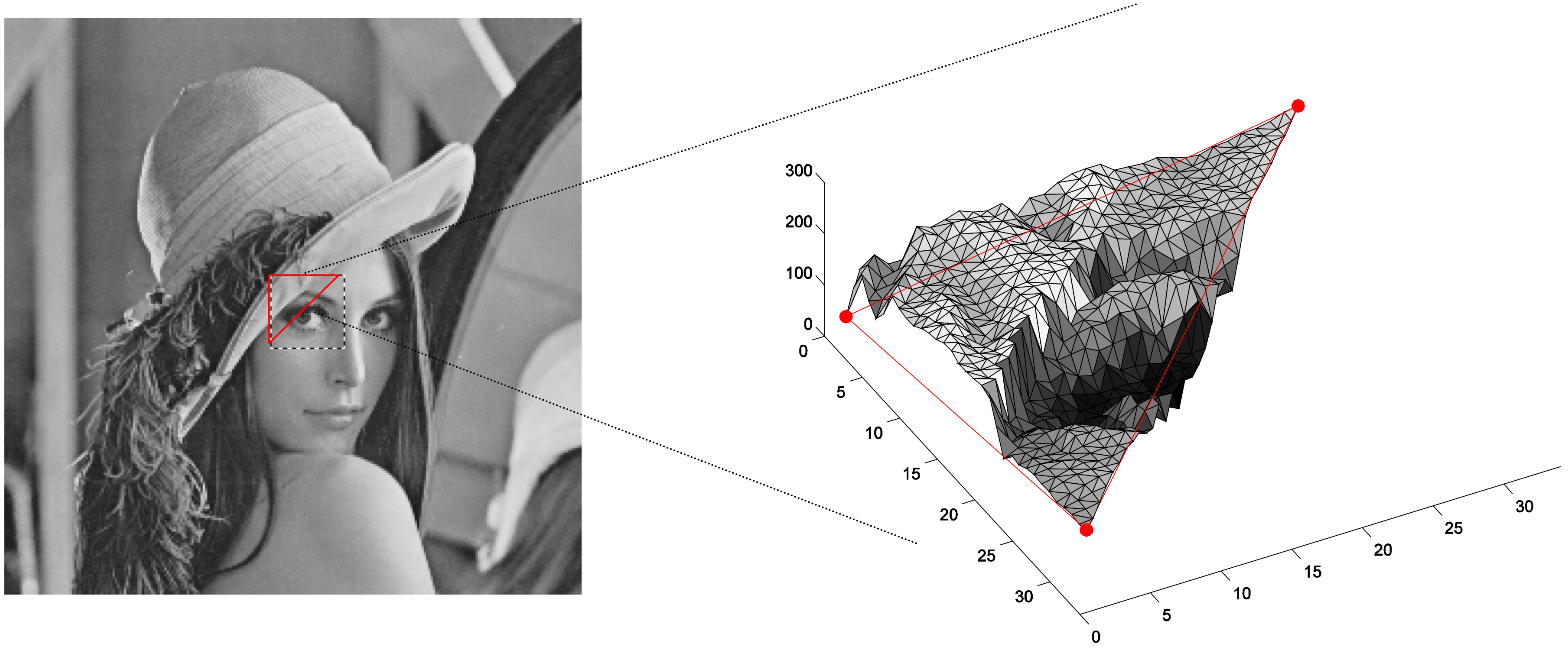


Domains for fractal interpolating surfaces over rectangular lattices using RIFS on (a) triangular tiling, (b) rectangular tiling.

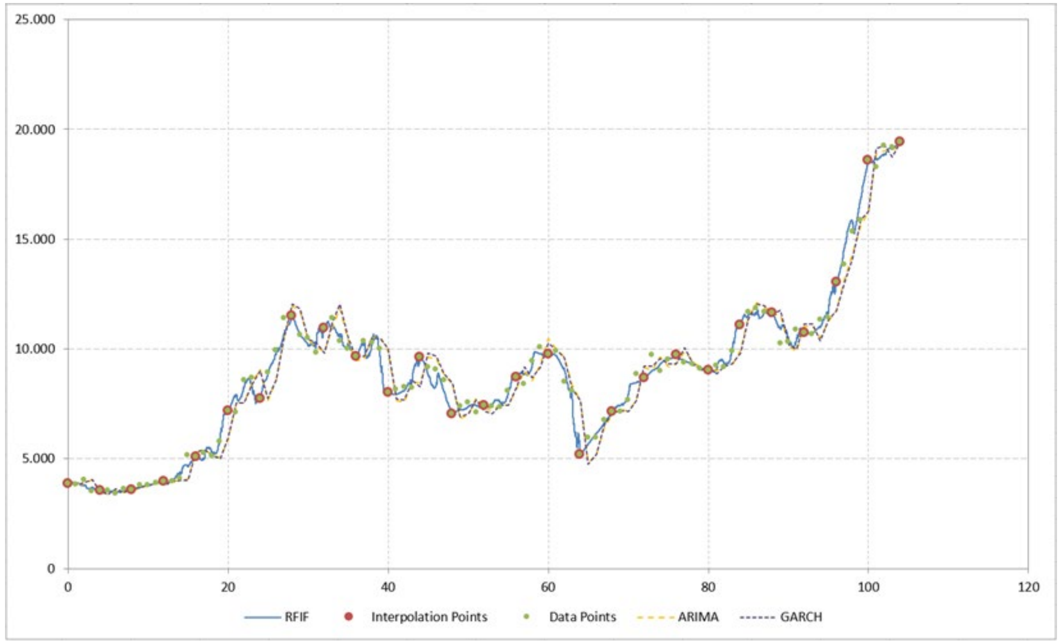
Rectangular tiling



Triangular tiling



Financial Time Series Modelling Using Fractal Interpolation Functions



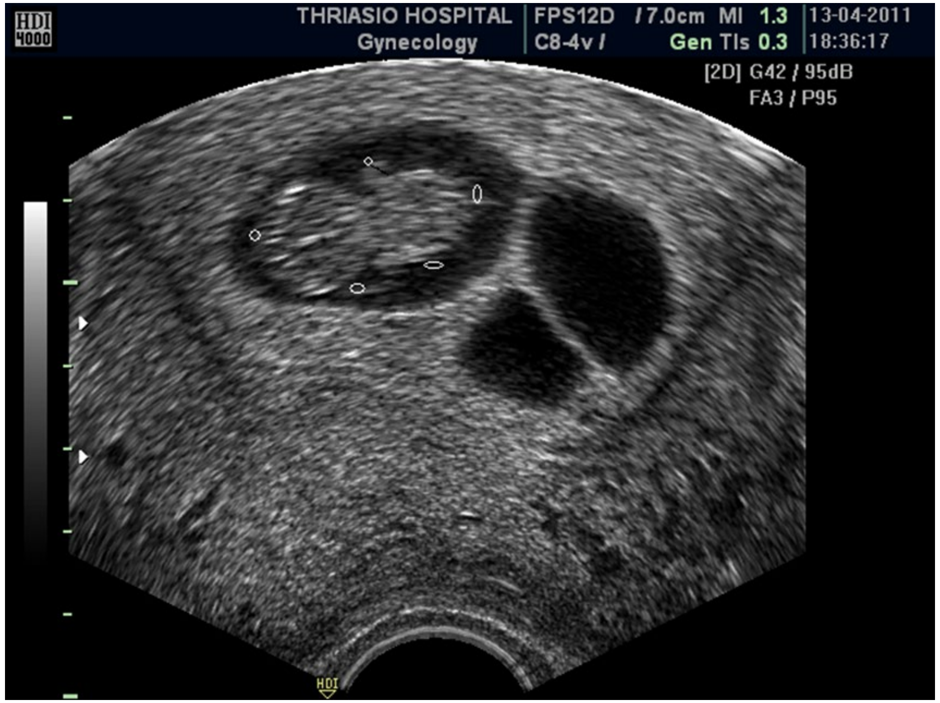
A time series of bitcoin prices (23 December 2018–16 December 2020) modelled by (i) a recurrent fractal interpolation function, (ii) the autoregressive integrated moving average ARIMA(1, 1, 0) model, (iii) the generalised autoregressive conditional heteroskedasticity GARCH(1,1) model.

Manousopoulos P., Drakopoulos V. and Polyzos E. (2023), AppliedMath 3(3), 510–524; https://doi.org/10.3390/appliedmath3030027

	Dataset 1—Bitcoin Prices		
	Mean Abs. Error	**Mean Abs. % Error**	**RMSE**
ARIMA	547.71	6.46%	740.16
GARCH	557.96	6.55%	762.67
RFIF	198.40	2.34%	289.18

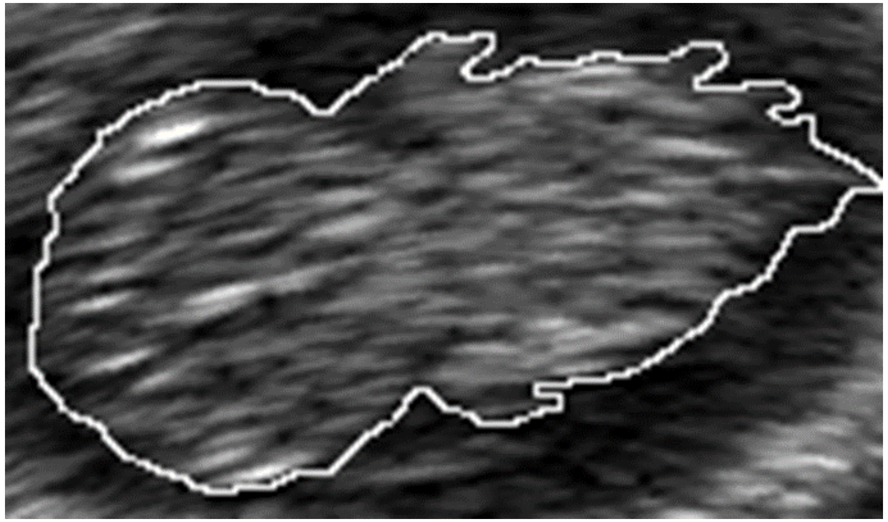
Results of the three methods for the first dataset (Bitcoin prices).

Obstetric ultrasound modelling and analysis with fractal interpolation methods



Eight-week ultrasound

Drakopoulos V. and Manousopoulos P. (2024) In Kevin Daimi, Abeer Alsadoon, and Sara Reis (eds.) Current and Future Trends in Health and Medical Informatics Studies in Computational Intelligence (SCI) Springer-Verlag.



Area of attachment of the trophoblast to the maternal wall

Outline

Outline

1. Prologue
2. Introduction
3. On the dimension
4. Iterated function systems
5. Fractal interpolation
6. Complex analytic dynamics

6. COMPLEX ANALYTIC DYNAMICS

- Preliminaries
- Dynamic spaces
- Parameter spaces

Holomorphic functions

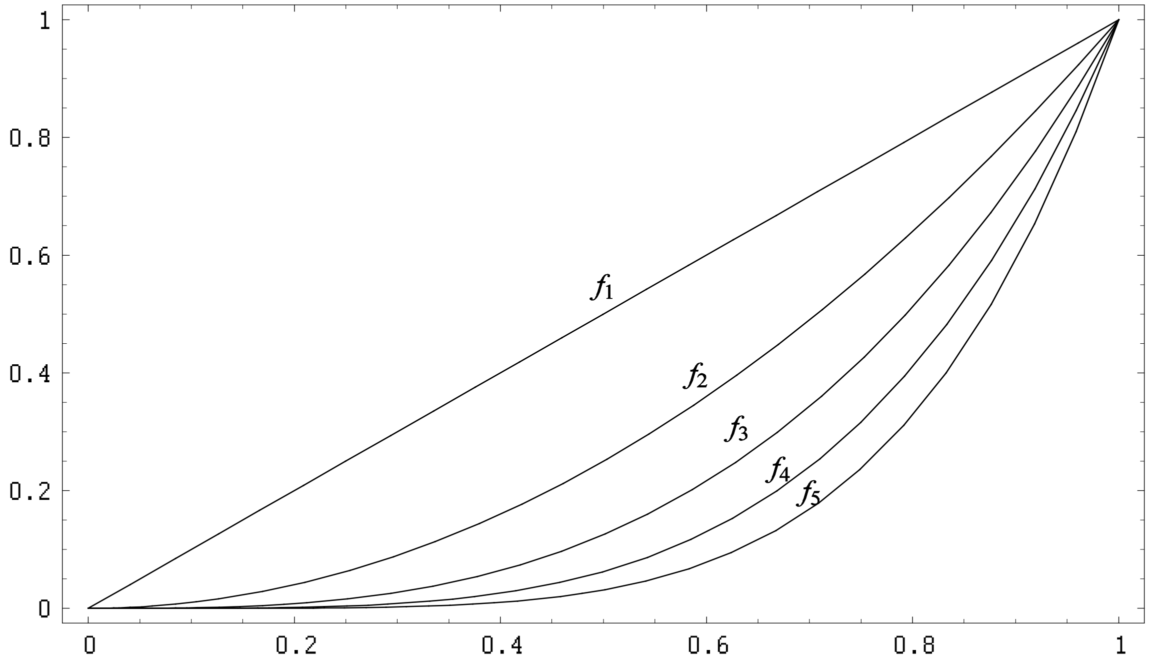
- A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighbourhood of each point in a domain in complex coordinate space $\mathbb{C}^n$.
- The existence of a complex derivative in a neighbourhood is a very strong condition: It implies that a holomorphic function is infinitely differentiable and locally equal to its own Taylor series (is *analytic*).
- Though the term *analytic function* is often used interchangeably with "holomorphic function", the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain.
- That all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

Convergence of functions

- Let X be a set and $f, f_n: X \to (Y, \rho)$, $n = 1, 2, \ldots$, where (Y, ρ) is a metric space.
- The sequence (f_n) converges uniformly to f if, for any $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and $n \geq n_0$.
- The sequence (f_n) converges locally uniformly on X to f if, every point $x \in X$ has a neighbourhood in which (f_n) converges uniformly to f .

Example

Let $f_n = x^n$, if $0 \le x \le 1$. The limiting function f has the value 0 in $[0, 1)$ and $f(1) = 1$. Since this is a sequence of continuous functions with discontinuous limit, the convergence is not uniform on $[0, 1]$.



Pointwise convergence

- If f is a polynomial of degree at least two, then for some radius r, we have
$$|f(z)| \geq 2|z|$$
on the open set $V = \{z : |z| > r\}$. Thus $f^n \to \infty$ uniformly on V. However, we also need the weaker convergence defined below.
- The sequence (f_n) converges pointwise to f if, for all $x \in X$,
$$\lim_{n\to\infty} f_n(x) = f(x),$$
i.e. for all $x \in X$ and $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon, x) \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) < \varepsilon$ for all $n \geq n_0$.

Convergence on subsets

- Let $U \subset \mathbb{C}$ and (f_n), f be functions defined on U taking values in $\mathbb{C}$.
- We say that the sequence (f_n) converges to f uniformly on all compact subsets of U, if, for any compact set $K \subset U$ and $\varepsilon > 0$, there is $n_0 = n_0(K, \varepsilon) \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq n_0$ and $z \in K$.
- Uniform convergence on compact subsets of U implies the pointwise convergence on all of U.

Example

- Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and f_n, f be complex functions with $f_n(z) = z^n$, $f(z) = 0$ for all $z \in U$ and $n = 1, 2, \ldots$.
- Then, for every $z \in U$, we have
$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} z^n = 0 = f(z),$$
and so $f_n \to f$ pointwise.
- Convergence of (f_n) is **not** uniform to f on U
- $f_n \to f$ uniformly on all compact subsets of U.

Normal families

- Let U be an open subset of $\mathbb{C}$ and $\mathcal{F}$ be a family of functions defined on U taking values in a metric space (Y, ρ).
- The family $\mathcal{F}$ is normal in U, if every sequence of functions from this family has a subsequence which converges uniformly on every compact subset of U.
- $\mathcal{F}$ is normal at a point z of U, if it is normal on some open set V containing z.
- Notice that $\mathcal{F}$ is then normal at every point of V.

Example

- Let $f(z) = \lambda z$, $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Thus $f^n(z) = \lambda^n z$. Let U be an open subset of $\mathbb{C}$. All depends on whether or not 0 is in U.
- $0 \notin U$. We claim that $\{f^n : U \to \mathbb{C}\}$ is normal. For, in any compact and hence closed subset V of U, there is a least value $r > 0$ of $|z|$ and so $|f^n(z)| \geq |\lambda^n| r$ for all z in V simultaneously. Thus $f^n \to \infty$ uniformly on V.
- $0 \in U$. The statement that $f^n \to \infty$ on all compact subsets V of U is no longer true, since $f^n(0) = 0$ and we can choose V to contain 0. Furthermore, $f^n \to f$ (f analytic) is ruled out because for any z in U we have $|f^n(z)| \to \infty$. Thus (f^n) is not normal on U.

Example

- Let $f(z) = \lambda z$, $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.
- Then (f^n) is normal on every open set U, because it converges uniformly to the constant function $\mathbf{0}$ on every compact set.

Uniformly boundedness

- A uniformly bounded family of functions is a family of bounded functions that can all be bounded by the same constant.
- This constant is larger than or equal to the absolute value of any value of any of the functions in the family.
- Every uniformly convergent sequence of bounded functions is uniformly bounded.
- The family of functions $f_n(x) = \sin nx$ travelling through the integers, is uniformly bounded by 1.

Montel's theorem

- The first, and simpler, version of the theorem states that a family of holomorphic functions defined on an open subset of the complex numbers is normal if and only if it is locally uniformly bounded.
- This theorem has the following formally stronger corollary. Suppose that $\mathcal{F}$ is a family of meromorphic functions on an open set D. If $z_0 \in D$ is such that $\mathcal{F}$ is not normal at z_0 and $U \subset D$ is a neighborhood of z_0, then

$$\bigcup_{f \in \mathcal{F}} f(U)$$

is dense in the complex plane.
- The stronger version of Montel's Theorem (occasionally referred to as the Fundamental Normality Test) states that a family of holomorphic functions, all of which omit the same two values $a, b \in \mathbb{C}$, is normal.

Julia and Fatou sets

- Around 1918–1920, the French mathematicians G. Julia and P. Fatou independently developed the theory of “Rational iteration” having Montel's Normality Criterion as their main tool.

Pierre Joseph Louis Fatou (1878–1929)

- They discovered the dichotomy of the Riemann sphere into sets that now bear their name.

Gaston Maurice Julia (1893–1978)

Julia and Fatou sets

- $J(f) = \{z \in \mathbb{C} : (f^n) \text{ is not normal at } z\}$
  $= \{z \in \mathbb{C} : (f^n) \text{ is normal on no open set containing } z\}$

- $F(f) = \mathbb{C} \setminus J(f)$
  $= \{z \in \mathbb{C} : (f^n) \text{ is normal on some open set containing } z\}$

Remarks

- $F(p)$ is open, so $J(p)$ is a closed subset of $\mathbb{C}$.
- In a previous example, case $0 \in U$, the sequence (f^n) was normal on no open set containing the origin 0. Thus $0 \in J(f)$.
- In the case $0 \notin U$ it was shown that if $z \neq o$, (f^n) is normal on some open set containing z, thus $z \in F(f)$.
- We conclude that $J(f) = \{0\}$ when $f(z) = \lambda z$ with $|\lambda| > 1$, a rather uninteresting Julia set. In case $|\lambda| < 1$, (f^n) is normal at every point of the plane, thus $J(f) = \varnothing$.
- Consequently, Julia sets of polynomials $f(z) = az + b$ will not be considered further.

Discrete dynamic systems

- For every $k \in \mathbb{N}$, we abbreviate as f^k the k-fold composition $f \circ f \circ \cdots \circ f$, where f^0 is the identity function.
- Let (X, ρ) be a metric space and let $f: X \to X$ be a transformation. An iterative scheme $\{f^k\}$ is called a discrete dynamic system.
- The forward orbit of a point $x \in X$ is the set

$$O^+(x) = \{f^n(x) : n \geq 0, f^0(x) = x\}.$$

Rational functions

- A rational map $R: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is of the form $R = P/Q$, where P and Q are polynomials without common factors and so without common roots.
- The degree of R is defined by

$$\deg(R)=\max\{\deg(P),\deg(Q)\}.$$

- How is $R(\infty)$ defined?

On quadratic polynomials

- A quadratic polynomial $q(z) = az^2 + 2bz + d$ ($a \neq 0$), where a, b, $d \in \mathbb{C}$ may be reduced by an affine change of coordinates $\Phi(z) = az + b$ ($a \neq 0$) to the form

$$p_c(z) = z^2 + c,$$

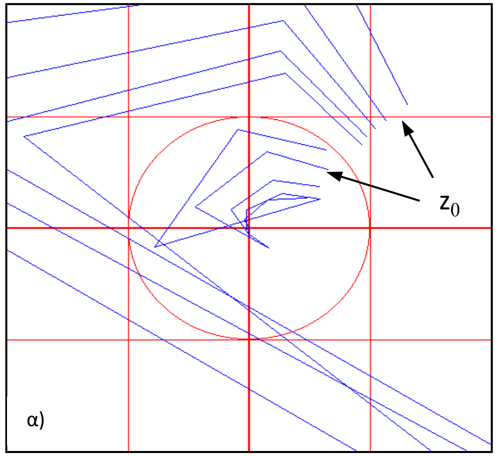
  where $c = ad - b^2 + b$.
- In order to understand the dynamics of all complex quadratic polynomials , it is enough to study the class of quadratic polynomials of the form $z \mapsto z^2 + c$, $c \in \mathbb{C}$.

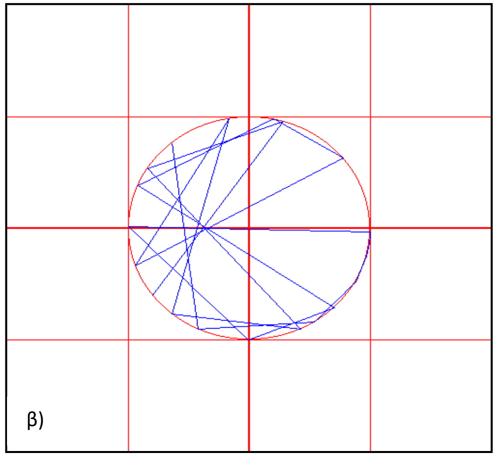
Example

- Let $f(z) = z^2$; then, for $z \in \mathbb{C}$,

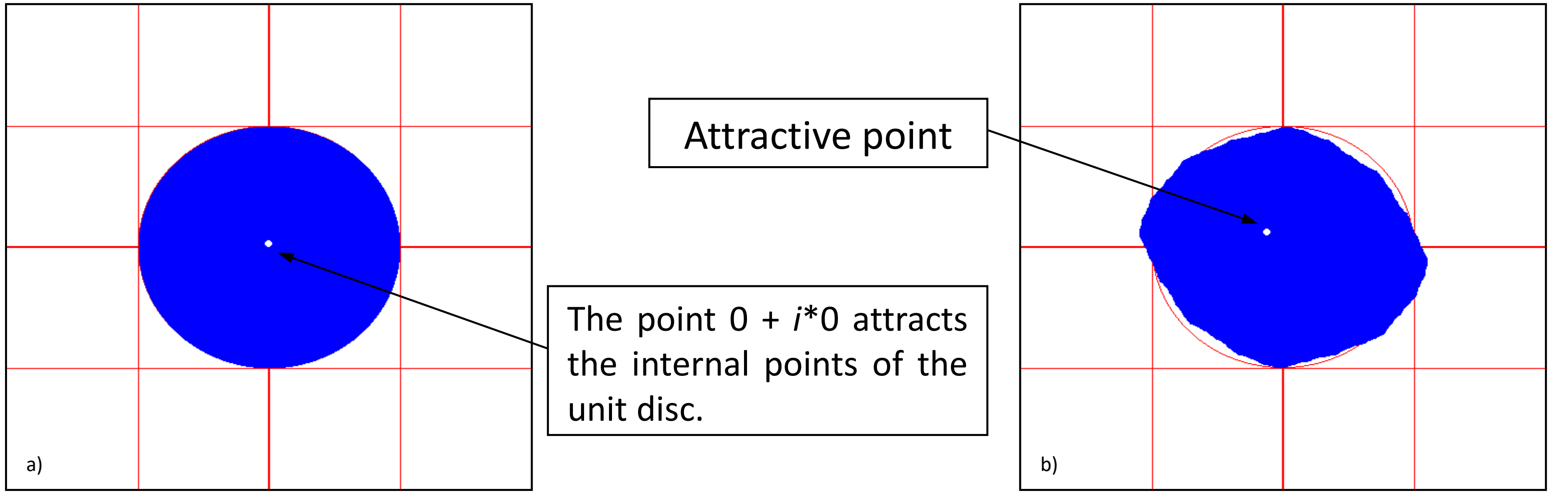
$$f^n(z) = z^{2^n}.$$

- $f^n(z)$ for $|z| > 1$ tends to infinity, whereas for $|z| < 1$ tends to zero.
- These two subsets of $\mathbb{C}$ are separated from the unit circle S^1.





Example



a) $c = 0 + i*0$, b) $c = -0.1 + i*0.1$.

Classification of periodic points

- A *fixed point* a of a rational function R is repulsive, indifferent or attractive depending on whether $|R'(a)|$ is greater than, equal to or less than one.
- $z = \infty$ is a (super)attracting fixed point of p.

Critical points and poles

- Critical values of a function R are those u for which $R(z) = u$ has a multiple root. Such solutions are called critical points of R and they are computed equivalently from $R'(z) = 0$.
- Let Ω be an open subset of $\mathbb{C}$, $a \in \Omega$ and $f\colon \Omega \setminus \{a\} \to \mathbb{C}$ be a holomorphic function. If

$$\lim_{z \to a} f(z) = \infty,$$

  then a is called a pole of f.
- $E_\lambda(z) = \lambda e^z$, $\lambda \in \mathbb{R}$ does not have critical points.

Theorem of Fatou

If $R(z)$ is a rational function having an attracting cycle, then at least one critical point will converge to it.

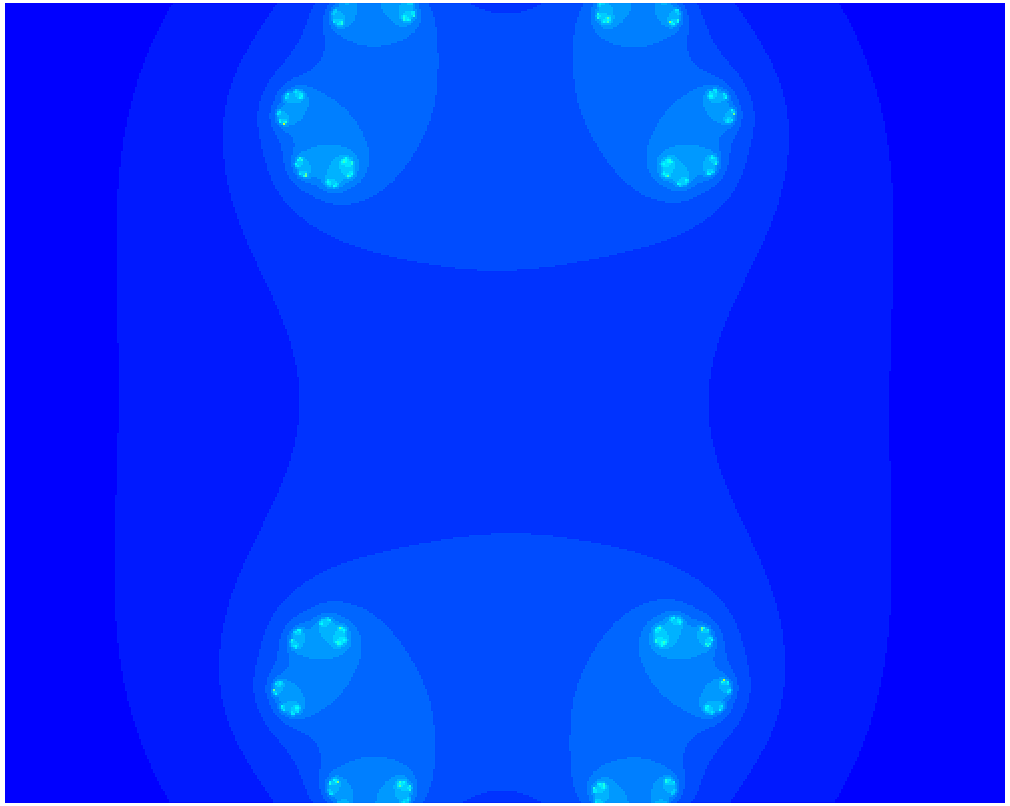
Quadratic polynomials

- The forward orbits of the critical points of a rational map determine the general features of the global dynamics of the map

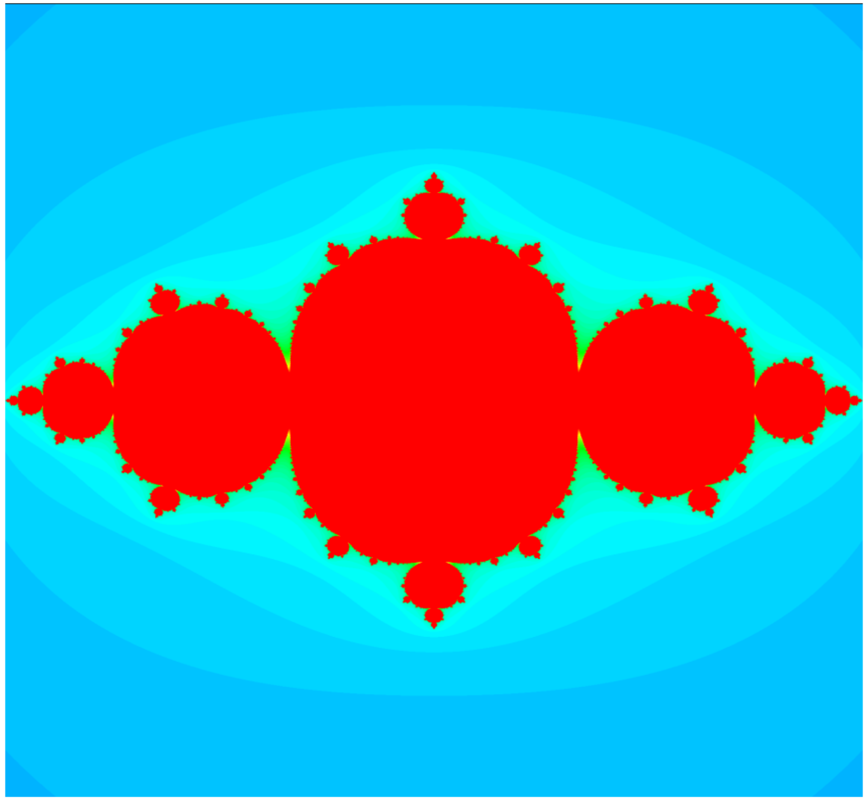
Julia sets of quadratic polynomials

- In what follows, $p_c(z) = z^2 + c$, where z, $c \in \mathbb{C}$ and $J(p_c) \equiv J_c$.
- p_c has at most one finite attractive fixed point or attractive cycle.
- If $\lim_{n\to\infty} p_c^n(0) \neq \infty$, then J_c is connected.
- If $\lim_{n\to\infty} p_c^n(0) = \infty$, then J_c is totally disconnected.

Julia sets for $z^2 + 0{,}7885\, e^{i\alpha}$, $\alpha \in [0, 2\pi)$



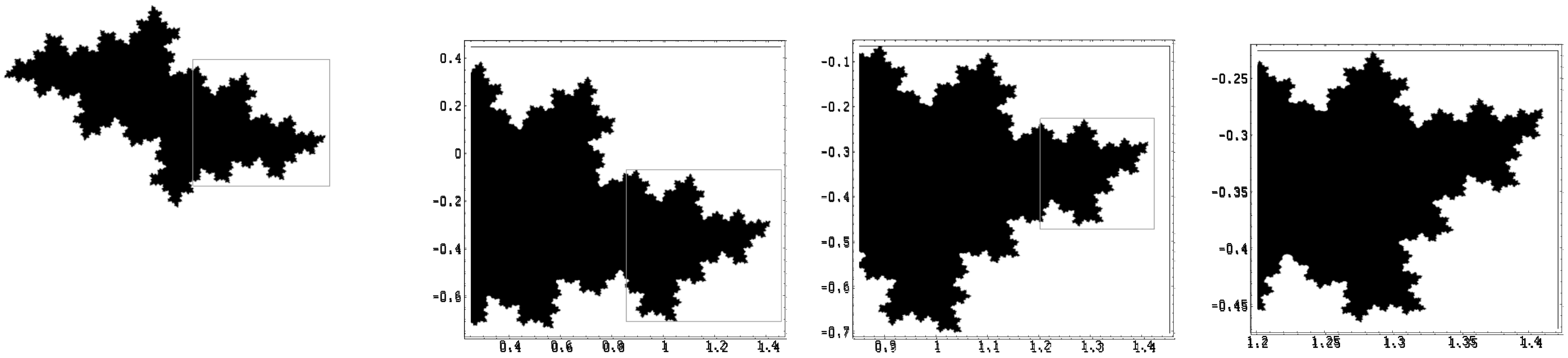
San Marco dragon



$c = -3/4$

Quasi self-similarity

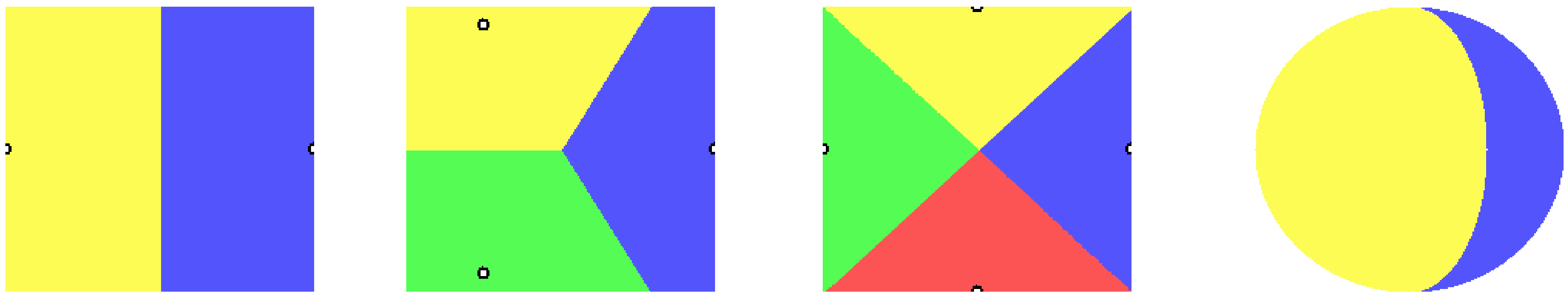
A looser form of self-similarity; the fractal appears approximately (but not exactly) identical at different scales.



$c = -0.5 + 0.5i$

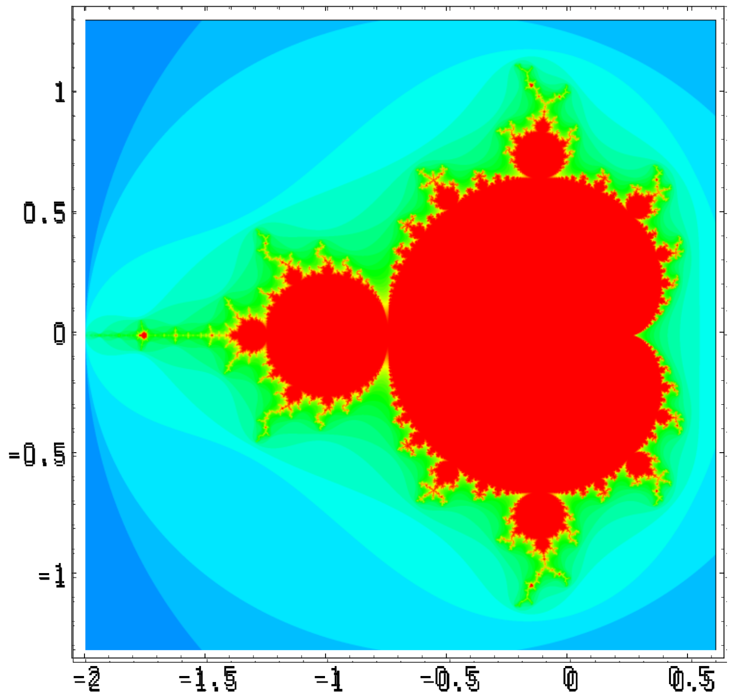
The basin

- The basin of attraction of an attractive fixed point a is the open set
$$A(a) = \left\{ z \in \overline{\mathbb{C}} \ : \lim_{k\to\infty} R^k(z) = a \right\}.$$
- The Julia and Fatou sets are $J(R) = \partial A(a)$ and $F(R) = \overline{\mathbb{C}} \setminus J(R)$, respectively.



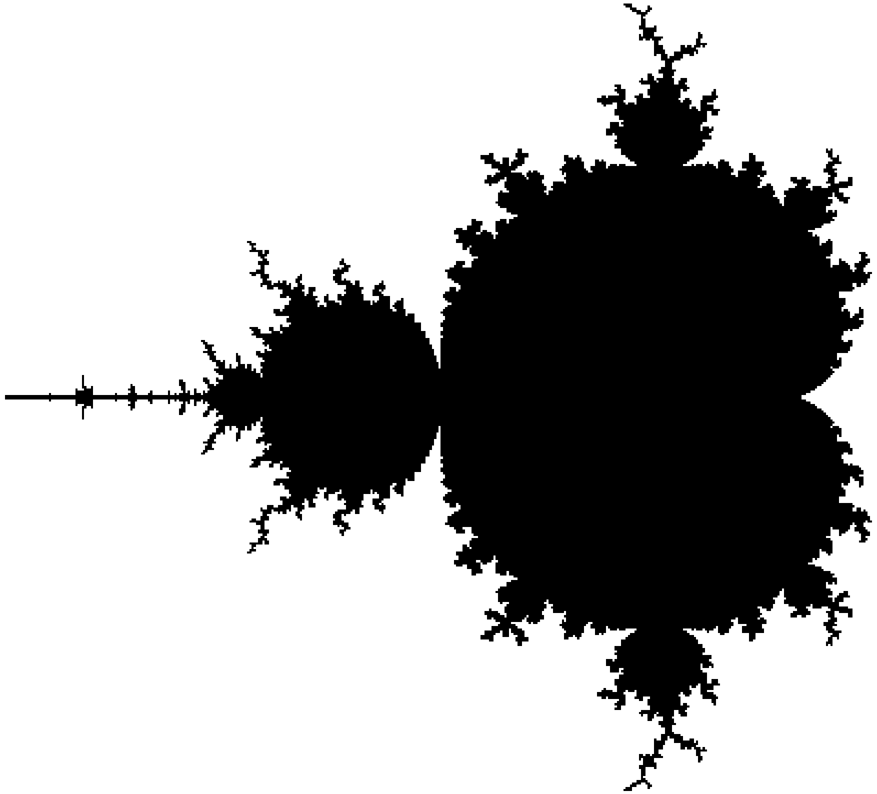
The Mandelbrot set

- The Mandelbrot set was originally defined as the set of points c for which the Julia set is connected.
- It comes as a surprise that this is exactly the set of points for which $p^n(0)$ does not tend to infinity as $n \to \infty$.



Douady-Hubbard

The $\mathcal{M}$ set is connected



Applications

- Biology
- Botany
- Chemistry
- Computer Science (Graphics, Vision, Image Processing and Synthesis)
- Geology
- Mathematics
- Medicine
- Physics

Fractal in Computer Graphics

- Fractal methods have proven useful for modelling a very wide variety of natural phenomena.
- In graphics applications, fractal representations are used to model terrain, clouds, water, trees and other plant life, feathers, fur, and various surface textures, and sometimes just to make pretty patterns.
- In other disciplines, fractal patterns have been found in the distribution of stars, river islands, and moon craters; in rain-field configurations; in stock market variations; in music; in traffic flow; in urban property utilisation; and in the boundaries of convergence regions for numerical-analysis techniques.

Classification

- Self-similar fractals
  - Have parts that are scaled-down versions of the entire object
  - Can use different scaling factors for different parts
  - *Statistically self-similar*, if random variations are applied
  - Commonly used to model trees, shrubs, plants
- Self-affine fractals
  - Have parts that are formed with different scaling parameters (*sx*, *sy* and *sz*) in different coordinate directions.
  - *Statistically self-affine*, if random variations are used
  - Commonly used to model terrain, water and clouds
- Invariant fractal sets
  - Formed with nonlinear transformations
  - *Self-squaring fractals,* e.g., the Mandelbrot set
  - *Self-inverse fractals*