On quadrirational pentagon maps [a]

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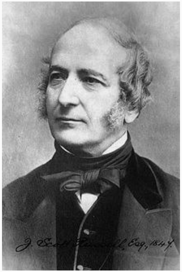
History/Introduction

John S. Russell and a “strange” wave



John Scott Russell 1808-1882

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

https://www.youtube.com/watch?v=wEbYELtGZwI

Joseph Boussinesq
1842-1928

Diederik Korteweg
1848-1941

Gustav de Vries 1866-1934

- Bussinesq (1877), Korteweg and de Vries (1895).

The KdV equation

$$u_t = u_{xxx} + 6uu_x$$

$u = u(x, t)$: displacement at position x at the time t

$$u_t = \frac{\partial u}{\partial t}, \qquad u_x = \frac{\partial u}{\partial x}, \qquad u_{xxx} = \frac{\partial^3 u}{\partial x^3}.$$

Zabusky and Kruskal (1965) Solitonic solutions of the KdV. Solitons

Martin Kruskal 1925-2006

Norman Zabusky 1929-2018

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$$u_t = u_{xxx} + 6uu_x, \quad (KdV)$$

- Assume that $u(x,t)$ is invariant under $\{x,t\} \to \{x - c\delta, t - \delta\}, \forall\delta$
  This solution depends on x, t necessarily as $\xi := x + ct$, and correspond to traveling waves (to the left) with constant speed c.
- The travelling wave solution

  $$u(x,t) = \phi(\xi), \quad \xi = x + ct,$$

  satisfies the nonlinear 1st order ODE

  $$\left(\frac{d\phi}{d\xi}\right)^2 = b + a\phi + c\phi^2 - 2\phi^3, \qquad (1)$$

- This ODE could be realized as the energy of a Hamiltonian system with cubic potential

$$E = \frac{1}{2}(\phi') + V(\phi), \quad V = \phi^3 - \frac{1}{2}(c\phi^2 + a\phi), \quad b = 2E.$$

- Assuming the boundary conditions $\phi, \frac{d\phi}{d\xi}, \frac{d^2\phi}{d\xi^2} \to 0$ when $|\xi| \to \infty$, we have $a = b = 0$ and (1) has as solution

$$u = \phi(\xi) = \frac{1}{2}c \; sech^2 \; \frac{1}{2}\sqrt{c}(x + ct + \delta), \quad sech\; x = \frac{1}{cosh\; x} = \frac{2}{e^x + e^{-x}},$$

- That is the wave of translations (soliton) that John Scott Russell observed
- The wave amplitude is exactly half of its speed c.

Integrable systems, and their dimensionality

- 3D Kadomstev-Petviashvili (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} = 0$$

- 2D Korteveg de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

- 1D Painlevé II (P_{II})

$$w_{xx} = 2w^3 + xw + \alpha$$

Nonlinear superposition of the solutions of the potential KdV equation

Nonlinear superposition of the solutions of the potential KdV equation[2]

- The potential KdV equation: $w_t = 6(w_x)^2 - w_{xxx}$.
- Appears as the compatibility conditions of the following system of Riccati equations[1]

$$\begin{aligned} w^{(1)}_t &= -w_t + 4[{k_1}^4 + {k_1}^2 w_x + (w_x)^2 + w_{xx}(w^{(1)} - w) + (w_x - {k_1}^2)(w^{(1)} - w)^2], \\ w^{(1)}_x &= -w_x - k_1^2 + (w^{(1)} - w)^2, \end{aligned} \tag{2}$$

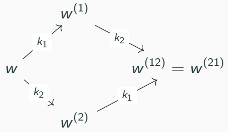
- Using $w_{xt} = w_{tx}$, $w^{(1)}_{xt} = w^{(1)}_{tx}$ we eliminate w to obtain the potential KdV equation expressed in $w^{(1)}$.
- The system (4) is called Bäcklund transformation with parameter k_1 for the potential KdV equation
- We obtain an one-parameter family of solutions $w^{(1)}$ of a given PDE from a given solution w

$$BT_{k_1} : w \overset{k_1}{\longmapsto} w^{(1)}$$

[1] Wahlquist, Estabrook, 1973

[2] is obtained from KdV by the substitution $u = -w_x$

- Starting with the initial solution w, we use the Bäcklund transformation and obtain the solutions $w^{(1)}$ and $w^{(2)}$ corresponding to parameters k_1 and k_2. For each of these solutions, we use the Bäcklund transformation (see figure) and require $w^{(12)} = w^{(21)}$. This will precisely determine the integration constants.



Bianchi commutativity diagram

- From the diagram we have the 4 Riccati

$$w_x^{(1)} + w_x = -k_1^2 + (w^{(1)} - w)^2, \qquad w_x^{(2)} + w_x = -k_2^2 + (w^{(2)} - w)^2,$$
$$w_x^{(12)} + w_x^{(1)} = -k_2^2 + (w^{(12)} - w^{(1)})^2, \quad w_x^{(12)} + w_x^{(2)} = -k_1^2 + (w^{(12)} - w^{(2)})^2$$

- We can obtain a purely algebraic relation

- We have

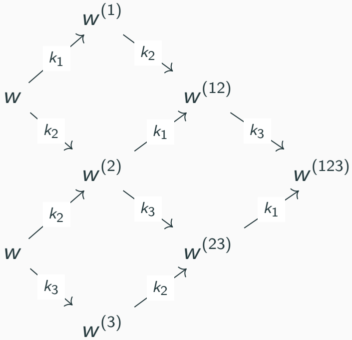
$$w_x^{(1)} - w_x^{(2)} = (w^{(1)} - w)^2 - (w^{(2)} - w)^2 + k_2^2 - k_1^2,$$
$$w_x^{(1)} - w_x^{(2)} = (w^{(12)} - w^{(1)})^2 - (w^{(12)} - w^{(2)})^2 + k_1^2 - k_2^2.$$

- By eliminating the derivatives, we obtain the so-called "Bianchi's permutability theorem," or, as it is otherwise known, the "nonlinear superposition of solutions" of the potential KdV equation

Nonlinear superposition principle of the potential KdV equation

$$(w^{(12)} - w)(w^{(1)} - w^{(2)}) = k_1^2 - k_2^2.$$

- So we can obtain the solution $w^{(12)}$ from $w, w^{(1)}, w^{(2)}$, purely algebraic!



Bianchi diagram of 3-soliton solution

- We can construct an infinite sequence of solutions to the potential KdV through the nonlinear superposition principle
- E.g.

$$
\begin{aligned}
w^{(123)} &= w^{(2)} + \frac{k_1^2 - k_3^2}{w^{(12)} - w^{(23)}} \\
&= \frac{k_1^2 w^{(1)}\left(w^{(2)} - w^{(3)}\right) + k_2^2 w^{(2)}\left(w^{(3)} - w^{(1)}\right) + k_3^2 w^{(3)}\left(w^{(1)} - w^{(2)}\right)}{k_1^2\left(w^{(2)} - w^{(3)}\right) + k_2^2\left(w^{(3)} - w^{(1)}\right) + k_3^2\left(w^{(1)} - w^{(2)}\right)}.
\end{aligned}
$$

The lattice potential KdV equation

From nonlinear superposition principle to the lattice potential KdV equation

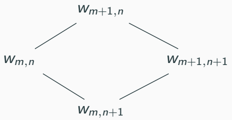
- Interpreting that the Bäcklund transformation introduces an additional discrete independent variable[3], i.e.

$$w_{m,n} := w, \quad w_{m+1,n} := w^{(1)}, \quad w_{m+1,n+1} := w^{(12)}, \quad etc., \quad m,n \in \mathbb{Z}$$

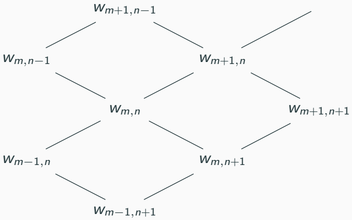
- The nonlinear superposition of solutions is reinterpreted as a discrete equation..

The lattice potential KdV equation

$$(w_{m+1,n+1} - w_{m,n})(w_{m+1,n} - w_{m,n+1}) = p_m^2 - q_n^2, \quad p_m := k_1, \quad q_n := k_2,$$



(a) The lattice potential KdV equation as an algebraic relation on a 2D-cell of the $\mathbb{Z}^2$ graph



(b) Fields in vertices of the $\mathbb{Z}^2$ graph

[3] Levi and Benguria 1980, Nijhoff, Quispel, and Capel 1983

- LpKdV is (alternating) translation invariant $w_{m,n} \mapsto w_{m,n} + (-1)^{m+n} c$, c a constant.
- LpKdV re-written in terms of the invariants

$$x_{m+1/2,n} := w_{m+1,n} + w_{m,n}, \quad y_{m,n+1/2} := w_{m,n+1} + w_{m,n},$$

The lattice potential KdV equation as an edge system

$$\begin{aligned} x_{m+1/2,n+1} &= y_{m,n+1/2} + \frac{k_1^2 - k_2^2}{x_{m+1/2,n} - y_{m,n+1/2}}, \\ y_{m++1,n+1/2} &= x_{m+1/2,n} + \frac{k_1^2 - k_2^2}{x_{m+1/2,n} - y_{m,n+1/2}}. \end{aligned}$$

- We can associate the map $R : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$

$$R : (x, y) \mapsto \left(y + \frac{k_1^2 - k_2^2}{x - y}, x + \frac{k_1^2 - k_2^2}{x - y} \right).$$

- R serves as solution of the *quantum Yang-Baxter equation*

The quantum Yang-Baxter equation

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$$

- Where $R_{ij} : X \times X \times X \mapsto X \times X \times X$, maps and X a set.
- The subscripts denote the sets where the map R acts non-trivially when is acting on $X \times X \times X$
- For example

$$R_{ij} : \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$$

$$R_{12} : (x, y, z) \mapsto \left(y + \frac{a_1 - a_2}{x - y}, x + \frac{a_1 - a_2}{x - y}, z \right),$$

$$R_{12} : (x, y, z) \mapsto \left(z + \frac{a_1 - a_3}{x - z}, y, x + \frac{a_1 - a_3}{x - z} \right),$$

$$R_{12} : (x, y, z) \mapsto \left(x, z + \frac{a_2 - a_3}{y - z}, y + \frac{a_2 - a_3}{y - z} \right).$$

Discrete traveling wave solutions[4]

- We consider functions $x = x_{m,n}$ on the $\mathbb{Z}^2$ graph with the periodicity

$$x_{m,n} = x_{m+d-1,n-1}, \qquad d \in \{2, 3, \dots, n\}.$$

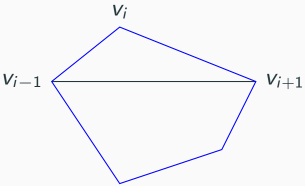
- Or equivalently functions invariant under $\{m, n\} \mapsto \{m + d - 1, n - 1\}$.
- Due to this periodicity, it follows that the dependent variable necessarily depends on m, n as $l := m + n(d - 1)$.
- So we have $x = x_l$. This reduction constitutes the discrete analog of the traveling wave reduction we saw for the KdV equation.
- Note: In the continuous case, this reduction yielded a family of ODEs of a fixed order. Here, we have a family of difference equations (ΔEs) whose order depends on the integer d.
- E.g. for $d = 3$, we have

$$y_{l+1} + y_l + y_{l-1} = \frac{a}{y_l} + b, \quad a = c_0 l + c_1 + c_2(-1)^l, \quad l \in \mathbb{Z},$$

The discrete Painlevé I.

[4] Papageorgiou, Nijhoff, Capel 1990

- Some integrable discrete dynamical systems can be constructed by repeating a specific geometric construction



[5] Adler 1993

Recutting of polygons [5] and the lattice potential KdV equation

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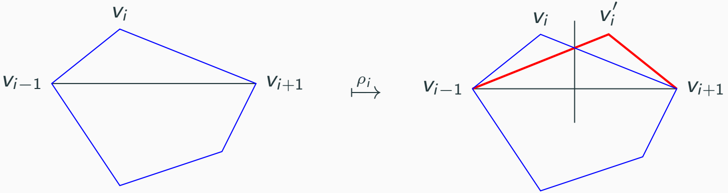
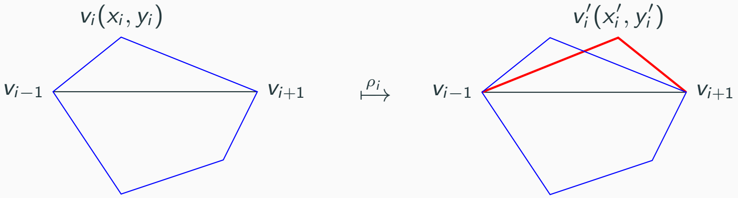


The recutting ρ_i moves the vertex v_i to v_i'. The other vertices are not affected

- $\rho_i^2 = id$ for any vertex v_i
- $\rho_i \circ \rho_j = \rho_j \circ \rho_i$ for any two non-consecutive vertices v_i, v_j
- $\rho_i \circ \rho_{i+1} \circ \rho_i = \rho_{i+1} \circ \rho_i \circ \rho_{i+1}$, for any two consecutive vertices v_i, v_{i+1}

[5] Adler 1993



- The angle $v_{i-1}\widehat{v_i}v_{i+1}$ and the area of the corresponding triangle is preserved after the recutting, so

$$\overrightarrow{v_{i-1}v_i} \cdot \overrightarrow{v_i v_{i+1}} = \overrightarrow{v_{i-1}v_i'} \cdot \overrightarrow{v_i' v_{i+1}}, \qquad \overrightarrow{v_{i-1}v_i} \times \overrightarrow{v_i v_{i+1}} = \overrightarrow{v_{i-1}v_i'} \times \overrightarrow{v_i' v_{i+1}}$$

- we obtain

$$\rho_i : \; u_i \mapsto u_i' = u_i + \frac{l_{i+1}^2 - l_i^2}{\bar{u}_{i+1} - \bar{u}_i},$$

where $u_i := x_i + \imath y_i$, $\imath^2 = -1$, $l_i^2 := (u_i - u_{i-1})(\bar{u}_i - \bar{u}_{i-1})$.

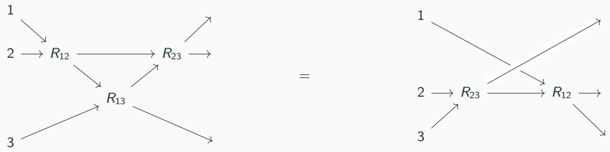
- It is related to lpKdV

• Part B: On quadrirational pentagon maps[a]

[a] C. Evripidou, P. K, A. Tongas, arXiv:2405.04945, 2024

The pentagon (or fussion) equation

- Moore, G. and Seiberg, N. 1989: Conformal field theory
- Maillet, J. 1990: 3-dimensional integrable systems



$$R_{12}R_{13}R_{23} = R_{23}R_{12},$$

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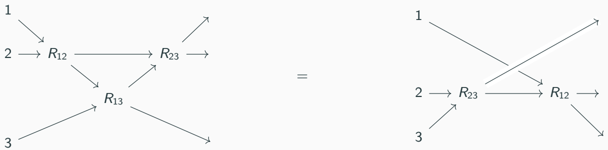

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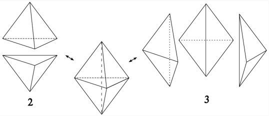

Pachner 2-3 move

- A set-theoretical version of the pentagon equation considers R_{ij} as maps $R_{ij} : X \times X \times X \to X \times X \times X$ where X is a set.

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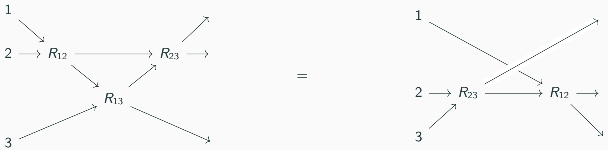

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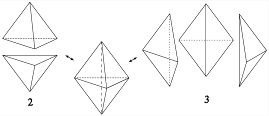

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The pentagon equation and pentagon maps

- Conformal field theory, Poisson maps, Hopf algebras, triangulations of piecewise linear 3-manifolds, Roger's dilogarithm, ... Discrete Integrable Systems
- Solutions of the set-theoretical version of the pentagon equation are called *pentagon maps*[6]

[6] Korepanov, Kashaev, Sergeev, Doliwa, Sharygin, Dimakis, Müller-Hoissen, Catino, Mazzotta, Miccoli, Colazzo, Jespers, Kubat

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Some discrete integrable systems can be naturally associated with pentagon maps.

- For example the pentagon map

$$S_l : (x, y) \mapsto (u, v) = \left(\frac{x}{x + y - xy}, x + y - xy \right)$$

is related to the Hirota-Miwa equation (discretization of KP)

$$\tau_{l+1,m,n}\tau_{l,m+1,n+1} + \tau_{l,m+1,n}\tau_{l+1,m,n+1} + \tau_{l,m,n+1}\tau_{l+1,m+1,n} = 0, \qquad l, m, n \in \mathbb{Z}$$

[6] Korepanov, Kashaev, Sergeev, Doliwa, Sharygin, Dimakis, Müller-Hoissen, Catino, Mazzotta, Miccoli, Colazzo, Jespers, Kubat

Equivalent forms of the pentagon equation

- The pentagon equation

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{12}$$

- The reverse (or dual) pentagon equation

$$S_{23} \circ S_{13} \circ S_{12} = S_{12} \circ S_{23}, \qquad S := \tau \circ R \circ \tau$$

- The braid-pentagon equation

$$B_{12} \circ B_{23} \circ B_{12} = B_{23} \circ \tau_{12} \circ B_{23}, \qquad B := R \circ \tau,$$

where $\tau : (x, y) \mapsto (y, x)$

- Mapping $S_I : (x, y) \mapsto (u, v)$ is equivalent to the refactorization problem

$$A(u)B(v) = B(y)A(x),$$

where the matrices A and B respectively read

$$A(x) := \begin{pmatrix} 1-x & x \\ 0 & 1 \end{pmatrix}, \qquad B(x) := \begin{pmatrix} 1 & 0 \\ 1-x & x \end{pmatrix},$$

- An alternative interpretation of the matrix refactorization problem is the following parameter dependent associativity condition (M.-Hoissen 2023)

$$p \circ_x (q \circ_y r) = (p \circ_u q) \circ_v r,$$

for p, q, r vectors in some vector space $\mathcal{V}$.

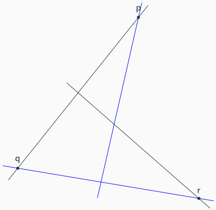
- Then the above associativity condition for the binary operation defined by

$$p \circ_x q := x\, p + (1-x)\, q,$$

delivers the map S_I.

• A geometric interpretation of the associativity condition is provided by a $(6_2, 4_3)$ configuration on the plane, the Veblen configuration

• The binary operation represents the collinearity of three points p, q, $p \circ_u q$



The Veblen configuration $(6_2, 4_3)$

• The pentagon equation reads as a consistency condition on the Desargues configuration (10_3) that contains five Menelaus configurations[7]

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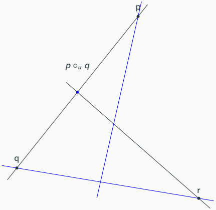

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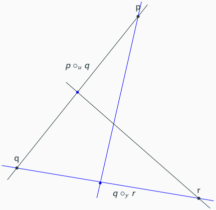

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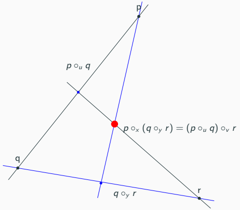

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[7] Doliwa, Sergeev 2014

A classification result[a]

[a] C. Evripidou, P. K, A. Tongas, arXiv:2405.04945, 2024

Equivalence relation

- There is a natural equivalence relation on pentagon maps

Two maps $R : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ and $S : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ are called *Möb* equivalent if there exists a bijection $\phi : \mathbb{X} \to \mathbb{X}$ such that $R \circ (\phi \times \phi) = (\phi \times \phi) \circ S$.

- The equivalence relation preserves the pentagon equation

Let $R : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ be a pentagon map and S a *Möb* equivalent map to R. Then S is also a pentagon map.

$$
\begin{aligned}
S_{12} \circ S_{13} \circ S_{23} &= (\phi^{-1} \times \phi^{-1} \times \phi^{-1}) \circ R_{12} \circ R_{13} \circ R_{23} \circ (\phi \times \phi \times \phi) \\
&= (\phi^{-1} \times \phi^{-1} \times \phi^{-1}) \circ R_{23} \circ R_{12} \circ (\phi \times \phi \times \phi) = S_{23} \circ S_{12},
\end{aligned}
$$

Quadrirational maps

- A map $R : \mathbb{X} \times \mathbb{X} \ni (x, y) \mapsto (u, v) \in \mathbb{X} \times \mathbb{X}$ is called quadrirational, if both the map R and the so called companion map (or partial inverse) $cR : \mathbb{X} \times \mathbb{X} \ni (x, v) \mapsto (u, y) \in \mathbb{X} \times \mathbb{X}$, are birational maps.

Said differently, the birational map $R = (u, v)$ is quadrirational if for any $y \in \mathbb{X}$ (generic), the map $u(., y) : x \mapsto u(x, y)$ is birational and for any $x \in \mathbb{X}$ (generic), $v(x, .) : y \mapsto v(x, y)$ is birational.

- In what follows $\mathbb{X} = \mathbb{CP}^1$, which we identify with $\mathbb{C} \cup \{\infty\}$ with its usual operations.

• Theorem[8]

Any quadrirational pentagon map $R : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$, with $R = (u, v)$ is *Möb* equivalent to exactly one of the following maps:

$$u = \frac{x}{x + y - xy}, \qquad v = x + y - xy, \qquad (S_I)$$

$$u = x, \qquad v = x + y - \delta xy, \qquad (S_{II}^{\delta})$$

$$u = \frac{x}{y}, \qquad v = y, \qquad (S_{III})$$

$$u = x - y, \qquad v = y, \qquad (S_{IV})$$

where $\delta = 0, 1$.

[8] C. Evripidou, P.K. and A. Tongas 2024

•Sketch of the proof

Let $R : (x, y) \mapsto (u(x, y), v(x, y))$, be a pentagon map

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{12} \, .$$

Then its components u, v necessarily satisfy the following relations

$$u(x, y) = u\left(u(x, v(y, z)), u(y, z)\right), \tag{3}$$

$$u(v(x, y), z) = v\left(u(x, v(y, z)), u(y, z)\right), \tag{4}$$

$$v(v(x, y), z) = v(x, v(y, z)). \tag{5}$$

We immediately recognize that (5) says that v is an associative function.

•[9] If $v : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ is a nonconstant associative rational function then there exists a Möbius transformation $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\phi^{-1} \circ v \circ (\phi \times \phi)$ is equal to $x, y, x + y$ or $x + y - xy$.

For any of the representatives (except $v(x, y) = x$, that does not give quadrirational maps) of associative rational functions above, we find all rational functions u that satisfy the equations (3) and (4).

Because of the quadrirationality of R, the rational function u is of the form $u(x, y) = \frac{a(y)x+b(y)}{c(y)x+d(y)}$, where the polynomials a, b, c and d are at most quadratic in y.

[9] J. V. Brawley, S. Gao, and D. Mills 2001

Discussion

- The inverse maps S_{I-IV}^{-1} of the Theorem satisfy the reverse pentagon equation, while the mappings $S_{I-IV} \circ \tau$ satisfy the braid-pentagon equation.
- The sets of singular points of the mappings S_{I-IV} respectively are

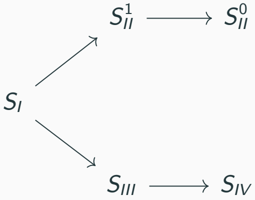
$$\Sigma_{S_I} = \{(0,0), (\infty, 1), (1, \infty)\}, \qquad \Sigma_{S_{II}^{\delta}} = \{(\infty, 1/\delta), (1/\delta, \infty)\},$$
$$\Sigma_{S_{III}} = \{(0,0), (\infty, \infty)\}, \qquad \Sigma_{S_{IV}} = \{(\infty, \infty)^2\}.$$

- The results of the theorem can be extended to the non-abelian setting

$$u = x(x + y - yx)^{-1}, \qquad v = x + y - yx, \qquad (\mathfrak{S}_I)$$
$$u = x, \qquad v = x + y - \delta yx, \qquad (\mathfrak{S}_{II}^{\delta})$$
$$u = xy^{-1}, \qquad v = y, \qquad (\mathfrak{S}_{III})$$
$$u = x - y, \qquad v = y, \qquad (\mathfrak{S}_{IV})$$

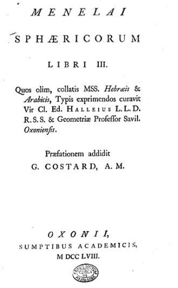
where $\delta = 0, 1$.

- Mapping S_I was firstly introduced in (Kashaev 1998) inside the context of quantum dilogarithm. Furthermore, S_I also results from the evolution of matrix KP solitons (Dimakis, Müller-Hoissen 2018). The non-abelian form of S_I that is $\mathfrak{S}_I$, arises as a reduction of the so-called *normalization map* (Doliwa, Sergeev 2014). Mapping $\mathfrak{S}_{II}^{\delta}$ (in an equivalent form) first appeared in (Kashaev, Sergeev 1998).
- There is the degeneration diagram



Degeneration diagram

Conclusions/lets go back in history (maybe 60-70 generations)



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Quos olim, collatis MSS. *Hebræis* & *Arabicis*, Typis exprimendos curavit Vir Cl. Ed. HALLEIUS L.L.D. R.S.S. & Geometriæ Professor Savil. *Oxoniensis*.

Præfationem addidit

G. COSTARD, A. M.

OXONII,

SUMPTIBUS ACADEMICIS,

M DCC LVIII.

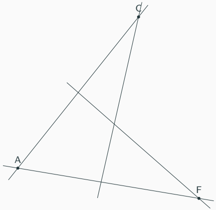
Μενέλαος ὁ Ἀλεξανδρεύς 70 - 140. Σφαιρικά

Menelaus theorem

- Let A, B, C be the vertices of a triangle and D, E, F be three points on the (extended) edges of the triangle opposite to A, B, C respectively. Then, the points D,E,F are collinear if and only if

$$\frac{\overline{AF}}{\overline{FB}}\,\frac{\overline{BD}}{\overline{DC}}\,\frac{\overline{CE}}{\overline{EA}} = -1,$$

where $\overline{PQ}/\overline{QR}$ denotes the ratio of directed lengths associated with any three collinear points P, Q, R.



The Veblen or Menelaus configuration $(6_2, 4_3)$

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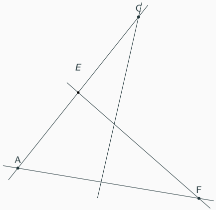

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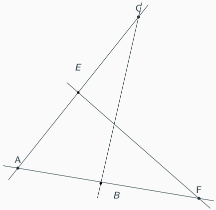

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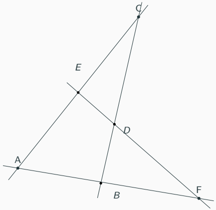

The Veblen or Menelaus configuration $(6_2, 4_3)$

Σας ευχαριστώ!