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Complex Behaviour in Classical and Quantum Chaos

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DYNAMICAL SYSTEMS AND COMPLEXITY

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ABSTRACT

I shall explain how chaos (chaotic behaviour) can emerge in deterministic systems of classical dynamics. It is due to the sensitive dependence on initial conditions, meaning that two nearby initial states of a system develop in time such that their positions (states) separate very fast (exponentially) in time. After a finite time (Lyapunov time) the accuracy of orbit characterizing the state of the system is entirely lost, the system could be in any allowed state. The system can be also ergodic, meaning that one single chaotic orbit describing the evolution of the system visits any other neighbourhood of all other states of the system.

In this sense, chaotic behaviour in time evolution does not exist in quantum mechanics. However, if we look at the structural and statistical properties of the quantum system, we do find clear analogies and relationships with the structures of the corresponding classical systems. This is manifested in the eigenstates and energy spectra of various quantum systems (mesoscopic solid state systems, molecules, atoms, nuclei, elementary particles) and other wave systems (electromagnetic, acoustic, elastic, seismic, water surface waves etc), which are observed in nature and in the experiments.

WHAT IS CHAOS IN A DETERMINISTIC DYNAMICAL SYSTEM?

By a **dynamical system** we mean a system whose state is defined by a point in the space of all possible states (**phase space**), and the motion (its evolution) in the past and in the future is entirely determined by **the local law of motion**.

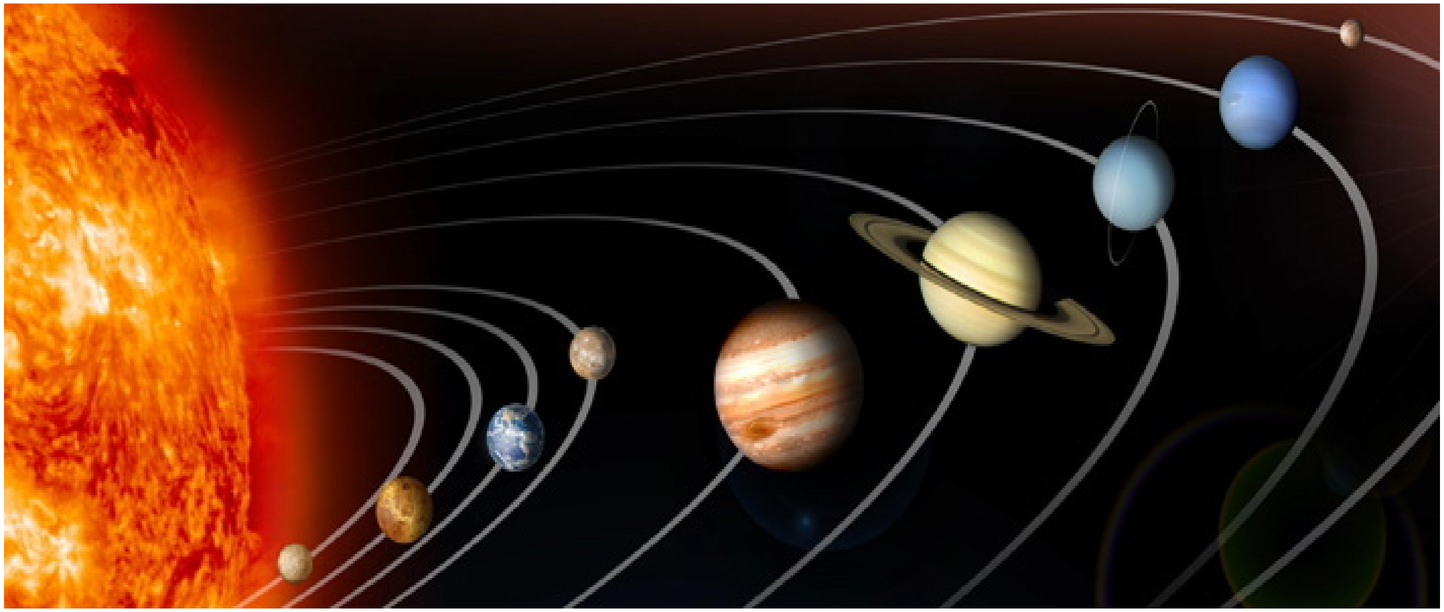
The law of motion can be either a differential equation or a difference equation. If such a law is known, and is exact, then the system is called **deterministic**.

EXAMPLE: The motion of a planet on a Kepler ellipse around the Sun, neglecting the influence of other planets, is a deterministic system, described by the Newtonian law of gravity for the attractive force, and the Newtonian equations of motion.

This is an example of a **stable and regular** deterministic system, which is **integrable**. It is a gravitational two-body problem.

If we switch on the attractive gravitational force by another large object (planet), the system is still entirely deterministic, but no longer integrable and regular: it is **nonintegrable** and **chaotic**. This is the so-called famous **gravitational three body problem (system)** proven to be chaotic by Henri Poincaré in the early 20th century.

The Solar System of 8 (or 9) planets (out of scale)



Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, (Pluto)

On the short time scale: the trajectories are ellipses (Kepler)

On the long run, the ellipses can stretch or shrink, rotate and tilt, even in a chaotic way.

The divergence/separation of nearby orbits in regular and chaotic systems are drastically different:

Slow: Linear in regular systems: Separation $\propto t$

Fast: Exponential in chaotic systems: Separation $\propto \exp(\gamma t)$

Lyapunov exponent $= \gamma$**, and Lyapunov time** $= \frac{1}{\gamma}$

For times much larger than Lyapunov time the motion is unpredictable

Example: In the case of Pluto: Lyapunov time $\approx$ **20 million years**

Wisdom and Susskind (1988) and Jacques Laskar (since 1990)

On the long run for certain initial conditions the planets and other bodies like asteroids and comets might move chaotically and collide with each other, or escape from the Solar System.

Nevertheless, in the case of Earth, we know that the motion is stable and regular, on an almost circular orbit, over about 5 billion years. This is crucial for the stability of climate and existence of life.

Motivation by the example of a simple dynamical system

Two-dimensional classical billiards:

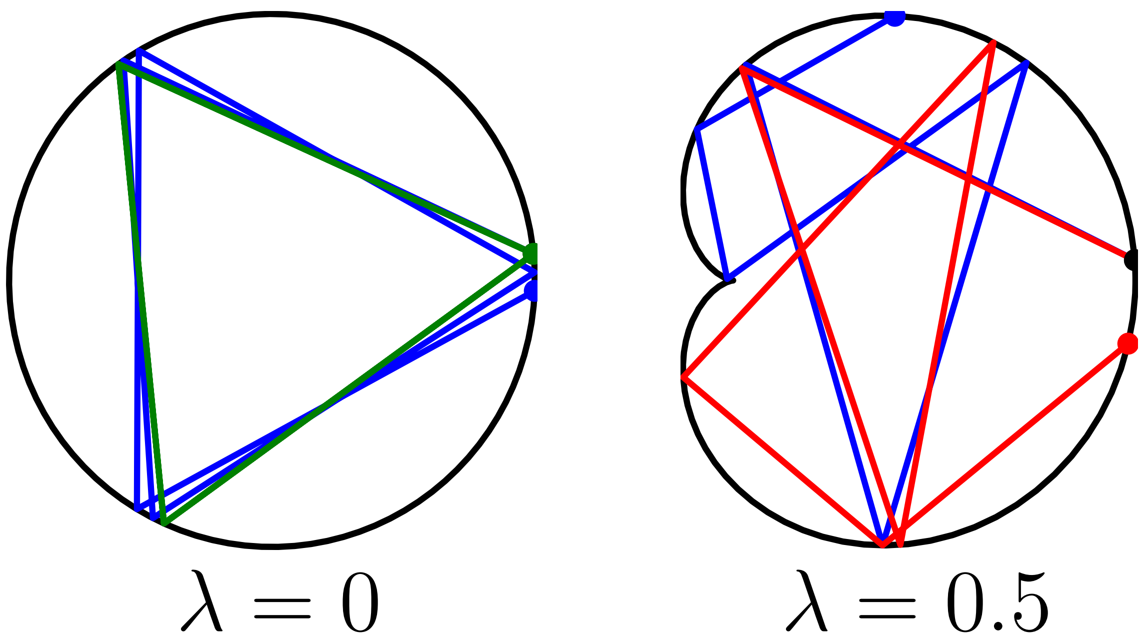
A point particle is moving freely inside a two-dimensional domain with specular reflection on the boundary upon the collision:

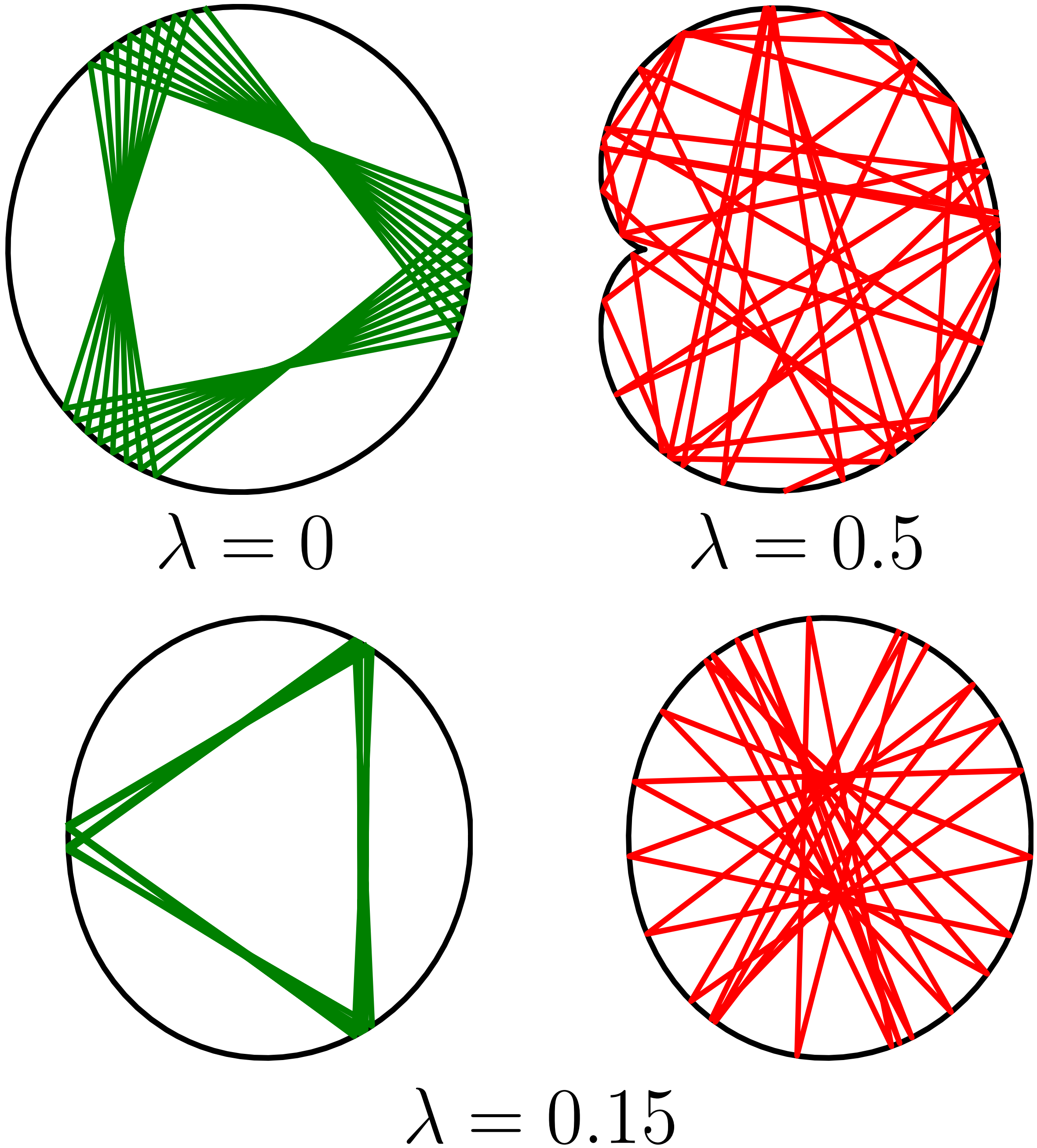
Energy (and the speed) of the particle is conserved.

A particular example of the billiard boundary shape as a model system:

Complex map: $z \to w$, $|z| = 1$

$$w = z + \lambda z^2,$$


$\lambda = 0$
$\lambda = 0.5$



$\lambda = 0$
$\lambda = 0.5$
$\lambda = 0.15$

QUANTUM CHAOS: Motivation by the simple example

Two-dimensional quantum billiards: a point particle trapped in a two-dimensional box

Helmholtz (Schrödinger) equation with Dirichlet boundary conditions

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + E\psi = 0$$

with $\psi = 0$ on the boundary

Solutions exist only at the discrete values of E: the eigenenergies.

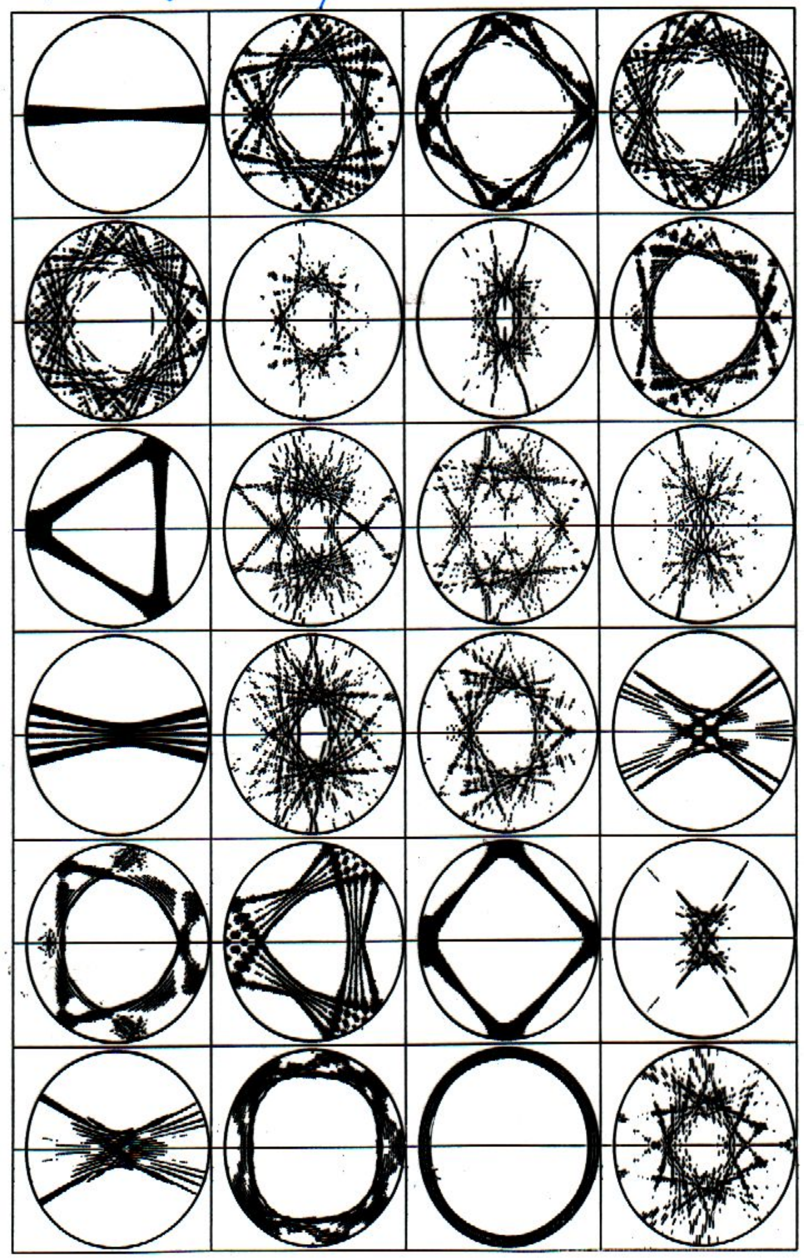
They are infinitely many, but countable.

This equation also describes the oscillations of an elastic membrane (a drum): E is the square of the eigenfrequency of the drum. ψ is the amplitude.

$W = z + \lambda z^2$ (Robnik 1983)

$\lambda = 0.15$

Figure 1



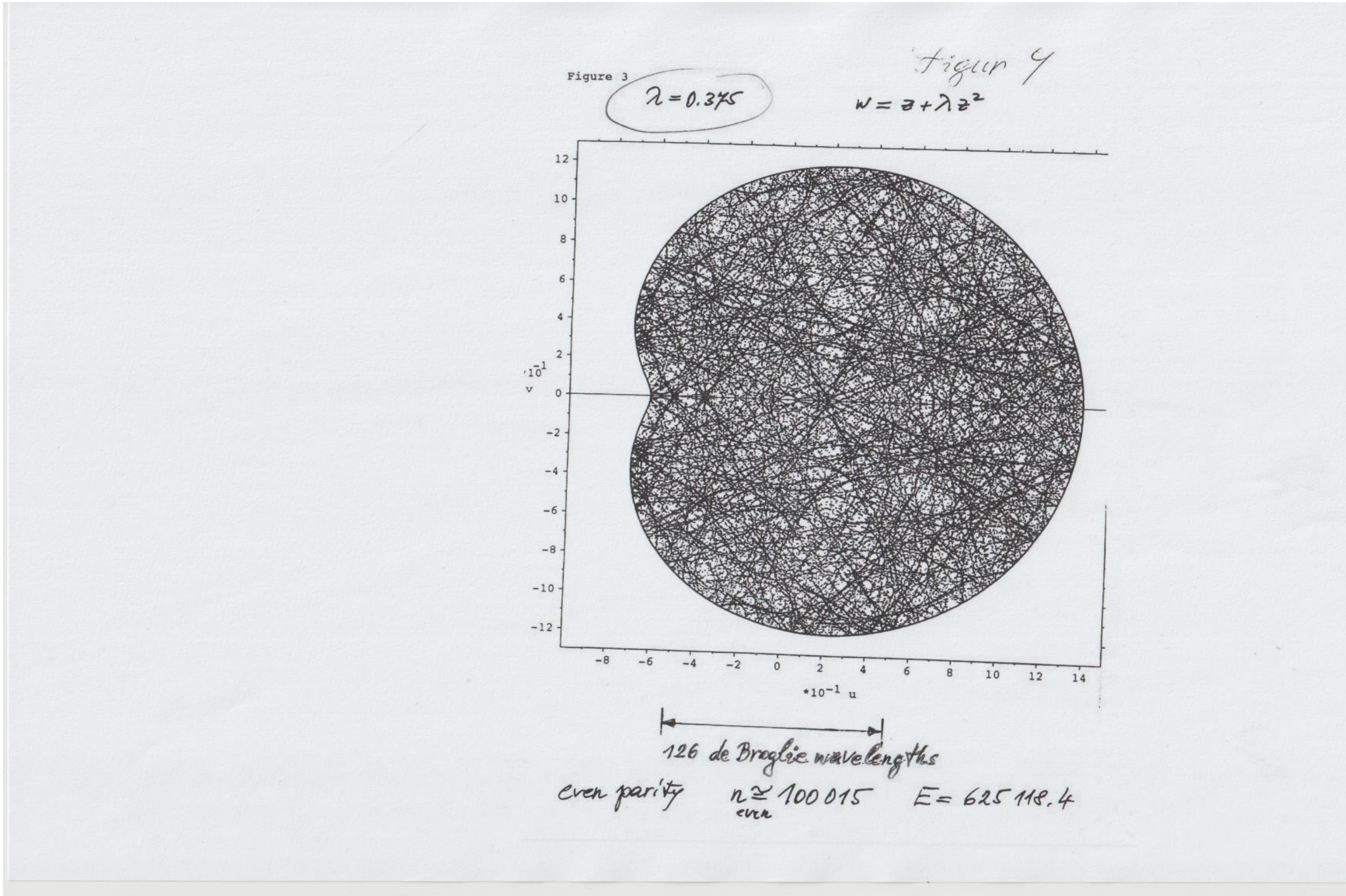


Figure 3
figur 4
λ = 0.375
w = z + λz²
*10⁻¹ v
*10⁻¹ u
126 de Broglie wavelengths
even parity
n ≅ 100 015
even
E = 625 118.4

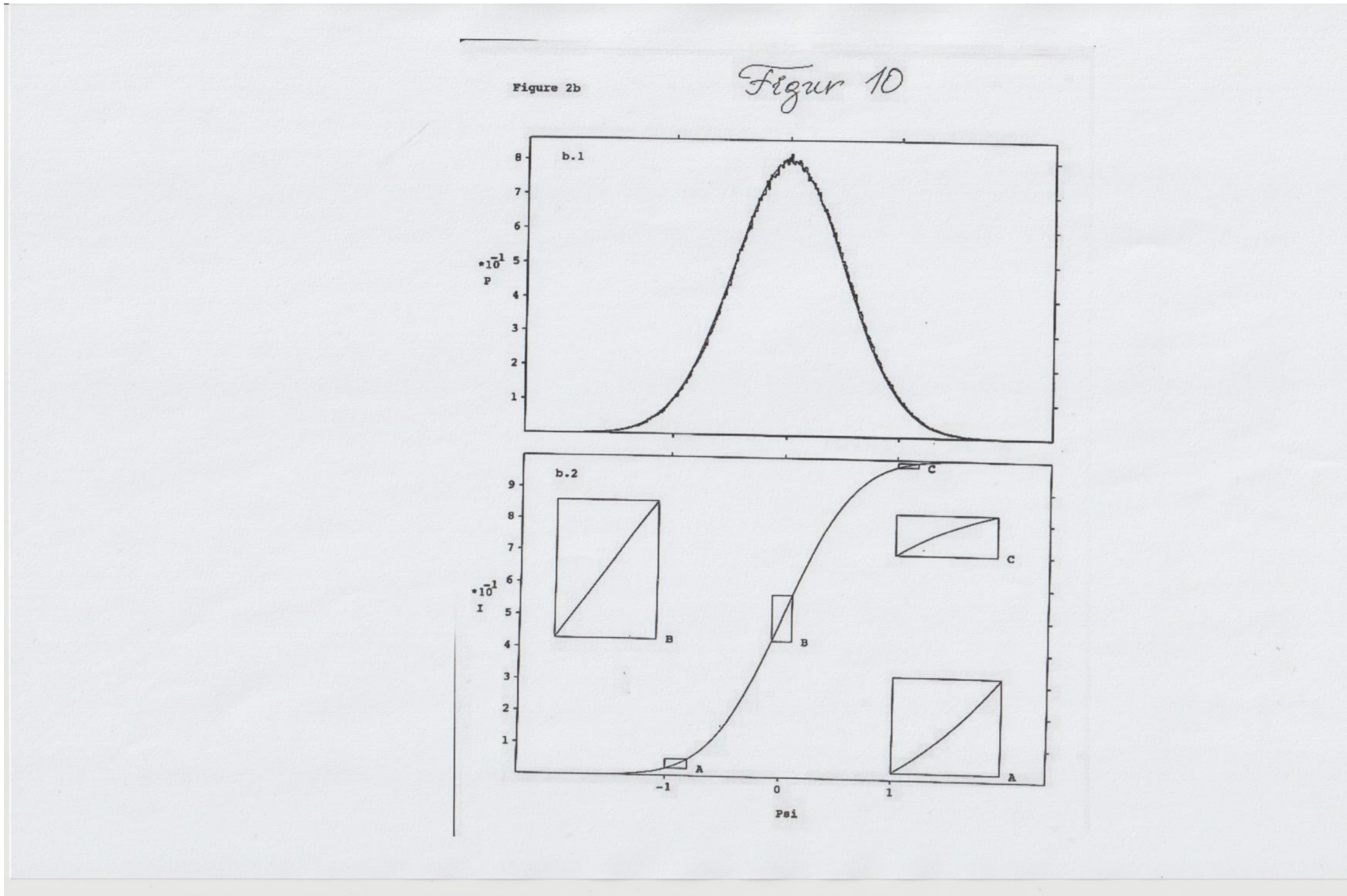


Figure 2b
Figur 10
b.1
$*10^{-1}$
P
b.2
$*10^{-1}$
I
A
B
C
Psi

Statistical properties of discrete energy spectra with the same density

Fig.I.8 - Segments of "spectra", each containing 50 levels. The "arrowheads" mark the occurrence of pairs of levels with spacings smaller than 1/4. See text for further explanation.

Bohigas and Giannoni 1984

One of the most important statistical properties:

Level spacing distribution $P(S)$
(after unfolding: the mean density equal to 1):

$P(S)dS =$ **probability that S is in the interval $[S, S + dS]$.**

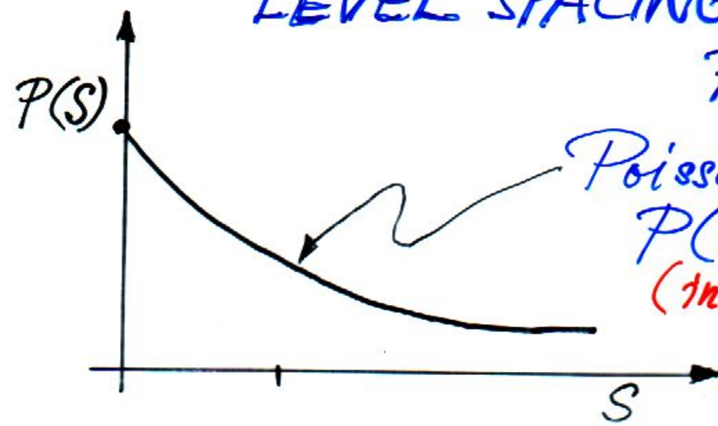
Bohigas - Giannoni - Schmit Conjecture (1984):

If the system is classically entirely chaotic, the statistical properties of the quantum energy spectra are well described by the statistical properties of the Gaussian random matrices:
GOE for real symmetric matrices and GUE for complex Hermitian matrices.

Meanwhile (after almost 25 years) the conjecture is proven by semiclassical methods using the Gutzwiller periodic orbit theory:

Universality of the classical unstable periodic orbits in fully chaotic systems implies universality of quantum spectral fluctuations.

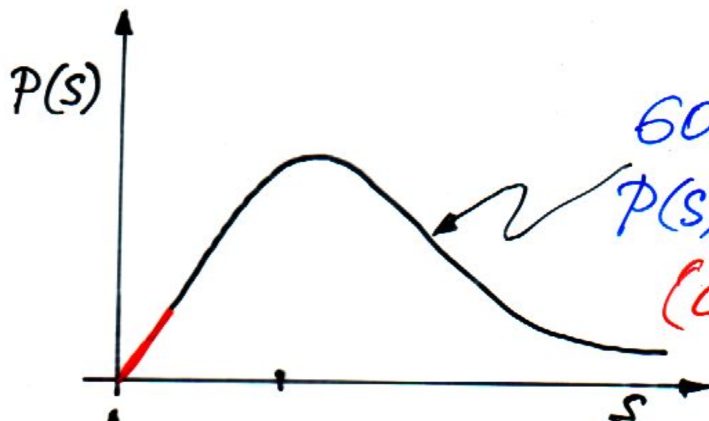
ENERGY (NEAREST NEIGHBOUR) LEVEL SPACING DISTRIBUTION $P(S)$



Poisson

$P(S) = e^{-S}$

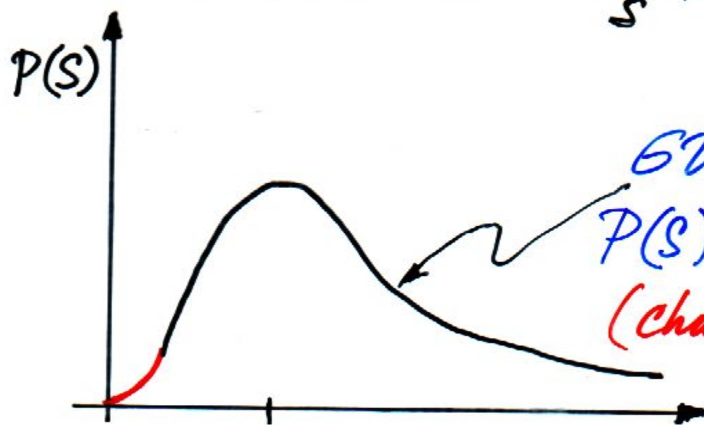
(integrability)



GOE ≈ Wigner

$P(S) = \frac{\pi S}{2} e^{-\frac{\pi S^2}{4}}$

(chaos & A.U.S.)



GUE ≈ Wigner

$P(S) = \frac{32 S^2}{\pi^2} e^{-\frac{4 S^2}{\pi}}$

(chaos, <u>no</u> A.U.S.)

If the system is of the mixed type, partially regular and partially chaotic, thus when regular and chaotic regions coexist in the classical phase space, we have semiclassical theories, which describe the intermediate region between the Poissonian and GOE/GUE statistics. They are well established and follow the so-called Principle of Uniform Semiclassical Condensation of Wigner functions in the quantum phase space (PUSC).

But let us look again at the classical mixed-type systems

integrable Hamiltonian systems:

N integrals (constants) of motion exist
N = number of degrees of freedom

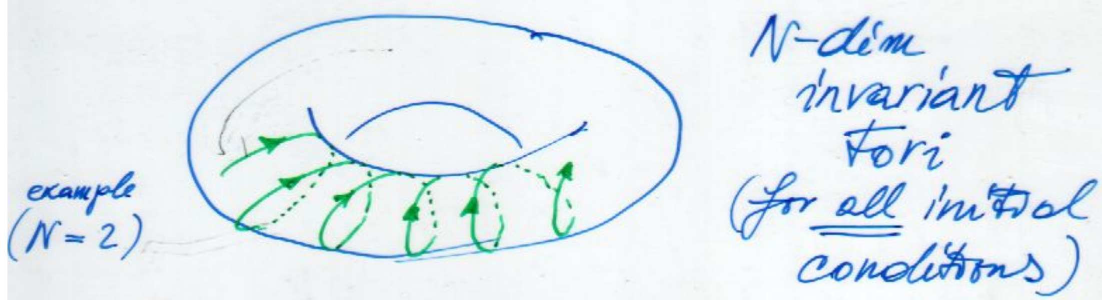
$$A_i = A_i(\vec{q}, \vec{p}) = A_i(\vec{q}(t), \vec{p}(t)) = \text{const.}$$

$$i = 1, 2, \ldots, N \qquad (A_1 = E = H(\vec{q}, \vec{p}))$$

$$\{A_i, A_j\} = \text{Poisson bracket} = 0, \ \forall i, j$$

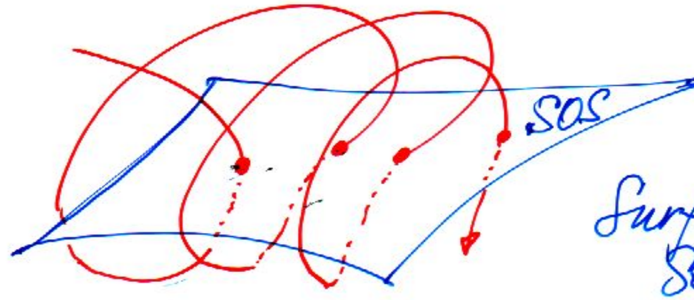
$$= \frac{\partial A_i}{\partial \vec{q}} \cdot \frac{\partial A_j}{\partial \vec{p}} - \frac{\partial A_i}{\partial \vec{p}} \cdot \frac{\partial A_j}{\partial \vec{q}} \equiv 0$$

Liouville-Arnold theorem:

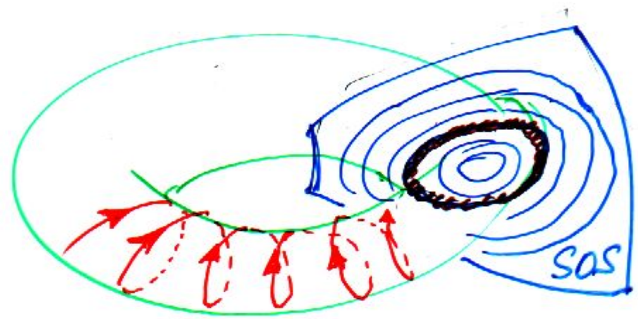


The ergodic systems (fully chaotic):

no integrals of motion except
the total energy $E = H(\vec{q}, \vec{p})$ - const.



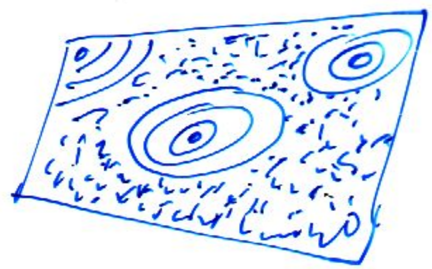
Surface of Section



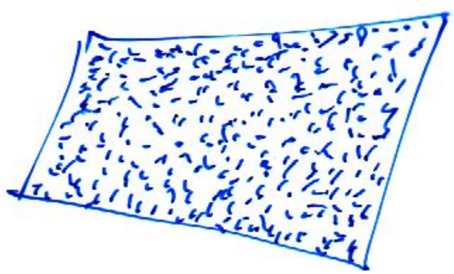
integrable:
N-tori
everywhere

perturb

Kolmogorov
Arnold
Moser



KAM
picture



ergodic
and
chaotic

Example of mixed type system: Hydrogen atom in strong magnetic field

$$H = \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} + \frac{eL_z}{2m_e c}|\mathbf{B}| + \frac{e^2\mathbf{B}^2}{8m_e c^2}\rho^2$$

B **= magnetic field strength vector pointing in** z**-direction**

$r = \sqrt{x^2+y^2+z^2}$ **= spherical radius,** $\rho = \sqrt{x^2+y^2}$ **= axial radius**

L_z **=** z**-component of angular momentum = conserved quantity**

Characteristic field strength: $B_0 = \frac{m_e^2 e^3 c}{\hbar^2} = 2.35 \times 10^9$ **Gauss =** 2.35×10^5 **Tesla**

Rough qualitative criterion for global chaos: magnetic force $\approx$ **Coulomb force**

(Wunner et al 1978+; Wintgen et al 1987+; Hasegawa, R. and Wunner 1989, Friedrich and Wintgen 1989; classical and quantum chaos: R. 1980+)

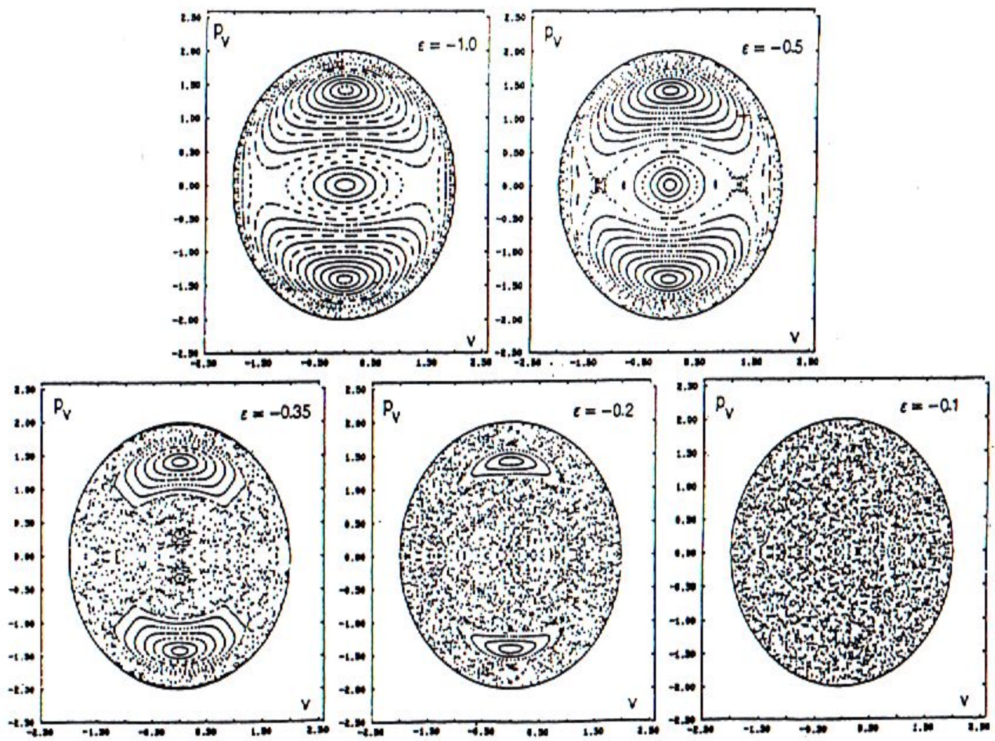


Fig. III-9. Poincaré surfaces of section $\Sigma(v, p_v;\ u=0)$ at different scaled energies (corresponding to increasing diamagnetic strength). The elliptic fixed point at the origin corresponds to the straight-line orbit I_∞, the other two fixed points to the straight-line orbit I_1.

CONCLUSION:

CLASSICAL CHAOS is characterized by the exponential divergence and sensitive dependence on initial conditions (positive Lyapunov exponent, $\gamma > 0$), and complex structure of the phase space.

Also, chaotic orbits/trajectories are characterized by the positive algorithmic complexity and are fundamentally unpredictable:

In order to predict a new segment of a trajectory one needs additional information proportional to the length of the segment itself and independent of the previous length of the trajectory. The information $I(t)$ associated with a segment of trajectory of length t is equal, for large t, to $I(t) = ht$, where h is the Kolmogorov-Sinai entropy, which is positive when $\gamma > 0$, positive Lyapunov exponent.

QUANTUM CHAOS comprizes phenomena in wave systems corresponding to the classical structures in the phase space, manifested in the short wavelength limit, and it also implies distinct statistical properties of the energy spectra associated with those structures.

Thank you very much!